MPT Functions and PDE Problems

These are the functions and PDE problems that are included in the MPT GUI.

1 1d Functions Menu

- step

$$f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases} \tag{1}$$

- x

$$f(x) = x \tag{2}$$

- W

$$f(x) = \begin{cases} 1 & -0.4 \leq x \leq -0.2 \\ \frac{15}{4}x + 1 & -0.2 \leq x < 0 \\ \frac{-15}{4}x + 1 & 0 \leq x \leq 0.2 \\ 1 & 0.2 \leq x \leq 0.4 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

- 2 jumps

$$f(x) = \begin{cases} \cos(\pi x/2) & x < -0.5 \\ x^3 - \sin(3\pi x/2) & -0.5 \leq x \leq 0.5 \\ x^2 + 4x^3 - 5x & x > 0.5 \end{cases} \tag{4}$$

- discontinuous sin

$$f(x) = \begin{cases} \sin \frac{\pi}{2}(x+1) & x \leq 0 \\ -\sin \frac{\pi}{2}(x+1) & x > 0 \end{cases} \tag{5}$$

- sharp peak center

$$f(x) = \begin{cases} -1 - x & x \leq 0 \\ (1-x)^6 & x > 0 \end{cases} \tag{6}$$

- smooth

$$f(x) = \exp[\cos(8x^3 + 1)] \tag{7}$$

- difficult test

$$f(x) = \begin{cases} \frac{1}{6}[G(x,\beta,z-\delta) + G(x,\beta,z+\delta) + 4G(x,\beta,z)] & -0.8 \leq x \leq -0.6 \\ 1 & -0.4 \leq x \leq -0.2 \\ 1 - |\,10(x-0.1)| & 0 \leq x \leq 0.2 \\ \frac{1}{6}[F(x,\alpha,a-\delta) + F(x,\alpha,a+\delta) + 4F(x,\alpha,a)] & 0.4 \leq x \leq 0.6 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where

$$G(x,\beta,z) = e^{-\beta(x-z)^2},$$

$$F(x,\alpha,a) = \sqrt{\max(1-\alpha^2(x-a)^2, 0)}$$

with $a = 0.5$, $z = -0.7$, $\delta = 0.005$, $\alpha = 10$, and $\beta = \frac{\log 2}{36\delta^2}$.

- saw tooth

$$f(x) = \begin{cases} x+1 & x<0 \\ x-1 & x \geq 0 \end{cases} \tag{9}$$

- center step

$$f(x) = \begin{cases} 0 & x < -0.4 \\ 1 & -0.4 \leq x \leq 0.4 \\ 0 & x > 0.4 \end{cases} \tag{10}$$

- sharp peak off-center

$$f(x) = \begin{cases} (2\exp[2\pi(x+1)] - 1 - \exp(\pi))/(\exp(\pi) - 1) & x < -0.5 \\ -\sin(2\pi x/3 + \pi/3) & x \geq -0.5 \end{cases} \tag{11}$$

- Shepp-Logan slice. Slice along constant $y = 0.5$ of the Modified Shepp-Logan phantom from Matlab Image Processing Toolbox.

- sin[cos(x)]*I[-0.5,0.5]

$$f(x) = \begin{cases} 0 & x < -0.5 \\ \sin[\cos(x)] & -0.5 \leq x < 0.5 \\ 0 & x \geq 0.5 \end{cases} \tag{12}$$

- discontinuous first derivative

$$f(x) = \begin{cases} -x & x < 0 \\ 2x & x \geq 0 \end{cases} \tag{13}$$

- discontinuous, periodic

$$f(x) = |\,x\,| \tag{14}$$

- analytic and periodic

$$f(x) = \exp[\sin(3\pi x) + \cos(\pi x)] \tag{15}$$

- discontinuity in f and f'

$$f(x) = \begin{cases} -x & x < 0 \\ 2x & 0 \leq x < 0.5 \\ 0 & x \geq 0.5 \end{cases} \tag{16}$$

2 2d Functions Menu

- square

$$f(x,y) = \begin{cases} 1 & |\,x| \leq 0.5 \text{ and } |\,y| \leq 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

- circular, non-constant

$$f(x,y) = \begin{cases} 3x + 2y^2 + 3 & (x^2 + y^2) < 0.25 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

- circular, constant

$$f(x,y) = \begin{cases} 1 & (x^2+y^2) < 0.25 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

- circular, non-compact support

$$f(x,y) = \begin{cases} 10x + 5 + xy + \cos(2\pi x^2) - \sin(2\pi x^2) & (x^2+y^2) \le 0.25 \\ 10x - 5 + xy + \cos(2\pi x^2) - \sin(2\pi x^2) & \text{otherwise} \end{cases} \tag{20}$$

- Shepp-Logan. Modified Shepp-Logan phantom from the Matlab Image Processing Toolbox.

- periodic, discontinuous

$$a(x,y) = \begin{cases} sin[\pi(x+y)] & |x| + |y| \le 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

$$b(x,y) = \begin{cases} sin[\pi(x+y)] & |x| + |y| > 0.75 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

$$f(x,y) = a(x,y) - b(x,y) \tag{23}$$

3 PDE Problems Menu

PDE solutions have been calculated with $N = 64, 128, 256, 512$

- Fourier - Linear Advection. The advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{24}$$

  with periodic boundary conditions.

  - dt = 0.001 (N=512), 0.005 (256,128), 0.01 (64), 4th order Runge-Kutta
  - T = 2.0

- Fourier - inviscid Burgers'. Inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{25}$$

  with periodic boundary conditions.

  - initial condition $u(x,0) = \sin[\pi(x+1)]$
  - stabilized with order $\rho = 16$ exponential filter
  - dt = 0.001
  - T = $2/\pi$

- Chebyshev - Linear Advection. Equation (25) with inflow/outflow boundary conditions.

- gird $x = \arcsin[\gamma \cos(\pi x)] / \arcsin(\gamma)$, $\gamma = 0.99$ (N=64, 128, 256), $\gamma = 0.999$ (512)
- dt = 0.01 (N=64), 0.005 (128), 0.002 (256), 0.001 (512)
- T = 2

• Chebyshev - Euler density. A spectral viscosity approximation of the density solution of Sod's problem for the Euler Equations of Gas Dynamics.

- T = 0.4
- N = 128; dt = 0.0005; gamma = 0.99; C = 1; s = 4;
- N = 256; dt = 0.0025; gamma = 0.999 C = 1; s = 4;
- N = 512; dt = 0.00025; gamma = 0.999; C = 1; s = 5;
- stabilized with the super spectral viscosity operator

$$\left(CN^{1-2s}\right)(-1)^{s+1}\left[\sqrt{1-x^2}\frac{\partial}{\partial x}\right]^{2s}\mathcal{I}_N u$$