A numerical study of a technique for shifting eigenvalues of radial basis function differentiation matrices

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Abstract

Radial Basis Function (RBF) collocation methods for time-dependent PDEs, in particular hyperbolic PDEs, are known to be difficult to implement in a way so that they are stable for time integration. It has been hypothesized that the instability is due to the way that boundary conditions are applied and to the relatively large errors in boundary regions. We describe a preconditioning technique that deemphasizes data in boundary regions and reformulates derivative calculations to focus on interior data. The preconditioning technique improves the eigenvalue stability of RBF methods for time-dependent PDEs. Unfortunately, the technique seems to only be applicable on domains that are simply shaped.

keywords: Radial Basis Functions, Numerical Partial Differential Equations, Time-Dependent Partial Differential Equations, Eigenvalue Stability, Collocation Methods.

1 Introduction

Eigenvalue stability of Radial Basis Function (RBF) methods for time-dependent PDEs have only recently started to be examined. In [15] it was proven that RBF methods are eigenvalue stable in the absence of boundaries, including methods on periodic domains. Additionally, special center distributions in 1d were found that caused the Gaussian RBF collocation method for PDEs with boundary conditions to be stable. The method was extended to the special case of tensor product grids in 2d. In [16], it was shown that eigenvalue stability could be obtained in the very general setting of complexly shaped domains and scattered centers by bounding the condition number of the system matrix of the RBF method. For a fixed set of centers, the shape parameter was selected so the the condition number of the system matrix did not exceed an upper bound. The approach was successful in obtaining eigenvalue stability when the scattered centers covered the domain in a fairly uniform manner. However, a stable condition number bound was not found in all cases, e.g. the case of widely scattered random center locations. Despite the condition number bound approach being successful in achieving eigenvalue stability in many cases, the approach has the negative consequence of that in order to achieve the bound, the shape parameter must be specified to be so large that the accuracy of the RBF method is severely degraded. Numerical experiments in [8] give further evidence that eigenvalue stability on a fixed set of somewhat uniformly spaced centers can be achieved by using a sufficiently large value of the shape parameter.

Ideally, RBF methods would be able to maintain eigenvalue stability when a shape parameter is used that allows the method to be most accurate. The RBF method is typically most accurate when its system matrix is "critically conditioned" [17]. With IEEE double precision floating point arithmetic, this is when the condition number of the system matrix is $\mathcal{O}(10e16)$.

2 RBF method for time-dependent PDEs

The RBF collocation method is based on differentiating a RBF interpolant. The method is sometimes called the asymmetric RBF collocation method, to distinguish it from the symmetric RBF collocation method [6], and may also be called Kansa's method [12]. The RBF interpolant on a set of centers

$\mathbf{x}_1^c, \ldots, \mathbf{x}_N^c$ in $\mathbb{R}^d$ is

$$s(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi(\left\|\mathbf{x} - \mathbf{x}_j^c\right\|_2, \varepsilon) \tag{1}$$

where

$$r = \|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}.$$

Many choices of the RBF $\phi(r, \varepsilon)$ exist. In all examples we use the multiquadric (MQ) RBF

$$\phi(r, \varepsilon) = \sqrt{1 + \varepsilon^2 r^2}$$

which is popular in applications and is representative of the class of RBF containing a free parameter called the shape parameter and which are theoretically spectrally accurate. The shape parameter controls both the accuracy and the conditioning of the method.

The coefficients, α, are determined by enforcing the interpolation condition

$$s(\mathbf{x}_i) = f(\mathbf{x}_i) \tag{2}$$

at a set of nodes that typically coincide with the centers. Enforcing the interpolation condition at N centers results in a $N \times N$ linear system

$$B\alpha = f \tag{3}$$

to be solved for the RBF expansion coefficients α. The matrix B with entries

$$b_{ij} = \phi(\left\|\mathbf{x}_i^c - \mathbf{x}_j^c\right\|_2), \qquad i, j = 1, \ldots, N \tag{4}$$

is called the system matrix. The uncertainty principle dictates that RBF methods can not be both accurate and well conditioned [18]. The conditioning of the RBF method is quantified by the matrix condition number, which in the 2 norm is

$$\kappa(B) = \|B\|_2 \left\|B^{-1}\right\|_2 = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{5}$$

where σ are the singular values of B.

To approximate the derivatives of a function $f(\mathbf{x})$ the RBF interpolant (1) is differentiated as

$$\frac{\partial}{\partial x_i} s(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \frac{\partial}{\partial x_i} \phi(\left\|\mathbf{x} - \mathbf{x}_j^c\right\|; \varepsilon). \tag{6}$$

Higher order derivatives, partial derivatives, and mixed partial derivatives, are handled in a similar manner. If (6) is evaluated at the centers $\left\{\mathbf{x}_j^c\right\}_{j=1}^N$ and written vector-matrix notation we have

$$\frac{\partial}{\partial x_i} s(\mathbf{x}) = \frac{\partial H}{\partial x_i} \alpha \tag{7}$$

where the evaluation matrix is the $N \times N$ matrix $\frac{\partial}{\partial x_i} H$ with entries

$$h_{ij} = \frac{\partial}{\partial x_i} \phi(\left\|\mathbf{x}_i^c - \mathbf{x}_j^c\right\|_2), \qquad i, j = 1, \ldots, N. \tag{8}$$

The notation H_{x_i} indicates a matrix with such entries. Likewise, $H_{x_i x_i}$ is used for second order derivatives.

From equation (3) and (7), the differentiation matrix

$$D_{x_i} = H_{x_i} B^{-1} \tag{9}$$

can be defined. We note that in (9) that D_{x_i} is formed without the inverse of B being explicitly calculated. The derivative of the function $f(\mathbf{x})$ at the centers $\left\{\mathbf{x}_j^c\right\}_{j=1}^N$ can be approximated by the single matrix multiplication

$$\frac{\partial}{\partial x_i} \approx \frac{\partial}{\partial x_i} s(\mathbf{x}) = D_{x_i} f.$$

The differentiation matrix is well-defined since it is known that the system matrix B is invertible for a large class of RBFs including the MQ [14]. The differentiation matrix may discretize a single space derivative or an entire differential operator. Recent books and monographs [3, 7, 17, 19] on RBF methods can be consulted for more detailed information.

After time-dependent PDEs are discretized in space with the RBF method, the remaining system of ODEs is advanced in time with an ODE method using a method of lines approach. A necessary condition for the method of lines to be stable is that the eigenvalues of the discretized spatial operator, scaled by the time step Δt, lie in the stability region of the ODE method. We have used a, four-stage, fourth-order, explicit Runge-Kutta (RK4) method in all numerical examples unless it is noted otherwise.

3 Preconditioning

The technique that is the basis of our preconditioning technique was developed for use with the Chebyshev pseudospectral (CPS) method. It was first used in [10] to improve the condition number of the Chebyshev pseudospectral method for second-order boundary value problems. Subsequently, is was used to improve accuracy, particularly in boundary regions, of the Chebyshev pseudospectral method in [2] and [5]. However, in neither work was the effect of the technique on eigenvalue stability examined. In a later section, we will briefly discuss the effect of preconditioning on eigenvalue stability of the CPS method.

It is well-known that errors from a RBF method will be the largest in boundary regions [9]. For example, see the left image of figure 2 and the accompanying discussion. One technique described in [9] to alleviate boundary errors is to cluster centers more densely in boundary regions similar to the way the standard Chebyshev pseudospectral grid does. However, we avoid boundary clustering in our examples in order to illustrate the power of the preconditioning technique and instead work with uniform center spacing in 1d and near uniform center spacing in 2d. We hypothesize that the relatively large errors near points where boundary conditions are applied are related to the instability issues with time integration for RBF methods for time-dependent PDEs. Next we summarize the method that is described in [2] and then in the numerical examples show how it can be modified and incorporated into the RBF collocation method.

The domain is taken to be the interval $[-1, 1]$ and the function $u(x, t)$ is assumed to be of the form

$$u(x,t) = \tau(x)w(x,t) + \frac{1}{2}(1+x)u(1) + \frac{1}{2}(1-x)u(-1). \tag{10}$$

Although we are only preconditioning space derivatives, the function is written as both a function of x and t as later it will be the solution of a time-dependent PDE. The preconditioned form is the product of a function $w(x, t)$ and a weight $\tau(x)$ that vanishes on the boundary, plus a linear interpolant between the boundary points. The equation

$$w(x,t) = \frac{u(x,t) - \frac{1}{2}\left[u(1) + u(-1) + x(u(1) - u(-1)\right]}{\tau(x)} \tag{11}$$

can be used to find $w(x, t)$ everywhere except at the boundary points. The reason that the linear term was added in equation (10) is so that at the

boundary points L'Hospital's rule is applicable and can be used to find that

$$w(\pm 1, t) = \pm \frac{\left(\frac{1}{2}(u(1) - u(-1)) - u_x(\pm 1, t)\right)}{\tau_x(\pm 1)}. \tag{12}$$

Then

$$u_x = \tau w_x + \tau_x w + \frac{1}{2}\left[u(1,t) - u(-1,t)\right] \tag{13}$$

where w_x is calculated by multiplication by the RBF differentiation matrix, i.e. $w_x = D_x w$. The weight function τ is small near the boundaries and damps out the relatively large errors in calculating w_x that occur near the boundaries. To calculate the second derivative, (13) is differentiated to get

$$u_{xx} = \tau w_{xx} + 2\tau_x w_x + \tau_{xx} w. \tag{14}$$

where $w_{xx} = D_{xx} w$. The weight function τ damps the boundary errors in w_{xx}, but the boundary errors in w_x are no longer damped as w_x is now multiplied by τ_x which is not necessarily small near the boundaries. However, in general the boundary errors in w_{xx} are much larger than in w_x and preconditioning the second derivative also results in better accuracy in boundary regions.

It is well-known that RBF methods do not differentiate constant exactly (unless a constant term is added to the expansion (1)). In the preconditioned RBF method, constants are differentiated exactly without the need to append an additional constant term to the expansion (1). In equation (11) we see that $w = 0$ if $u(x)$ is a constant function, and that its first and second derivative are then exactly calculated by (13) and (14).

4 Numerical Examples

4.1 Advection equation with zero Dirichlet BC

First we consider the advection equation

$$u_t - u_x = 0 \tag{15}$$

on the interval $\Omega = [-1, 1]$ with the boundary condition $u(1, t) = 0$. The initial condition for the example is $u(0, t) = \exp(-40(x - 0.4)^2)$. The zero Dirichlet boundary condition at only one end of interval and the fact that

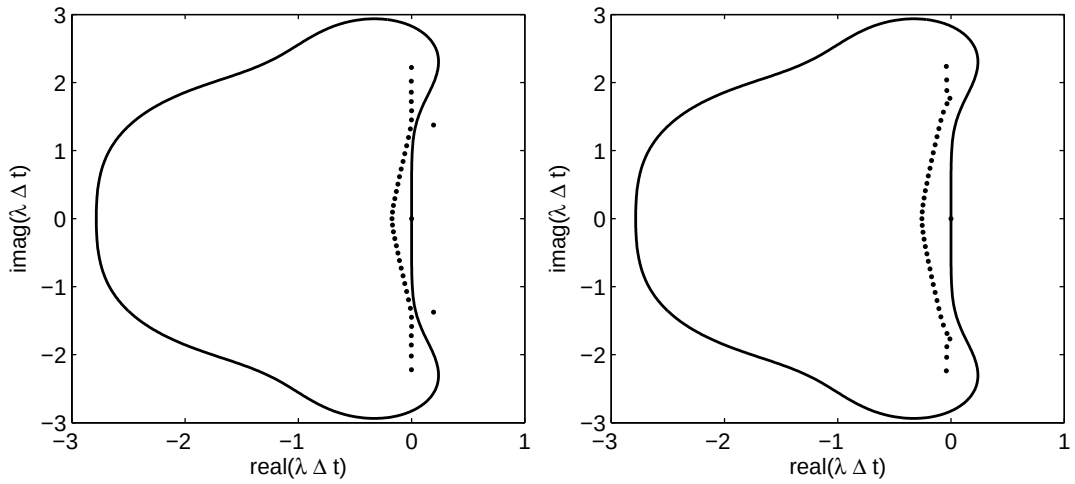

Figure 1: Differentiation matrix eigenvalues scaled by $\Delta t = 0.04$ in the RK4 stability region with $N = 40$ evenly spaced centers and $\varepsilon = 1.8$. Left: standard implementation differentiation matrix with $\max(\text{real}(\lambda)) = 4.87$. Right: Precondition differentiation matrix spectrum. All λ have non-positive real parts.

$u_x(1,t)$ is negligible with respect to machine precision at the boundary reduces (10) to

$$u(x,t) = \tau(x)w(x,t). \tag{16}$$

In order for the PDE in the variable w to have boundary conditions that make the problem well-posed, the weight function must only be zero at points were boundary conditions are applied. Thus for this problem, the weight $\tau(x) = x - 1$ is used so that in calculating the derivative

$$u_x = (x-1)D_x w + w$$

the information near the point where the boundary condition is enforced is deemphasized. To examine the stability of the preconditioned method, we see that in the variable w, the PDE (15) becomes

$$w_t = w_x + \left(\frac{\tau_x}{\tau}\right) w \tag{17}$$

with boundary condition $w(1,t) = 0$. The differentiation matrix for (17) is

$$\bar{D} = D_x + \text{diag}\left(\frac{1}{x-1}\right). \tag{18}$$

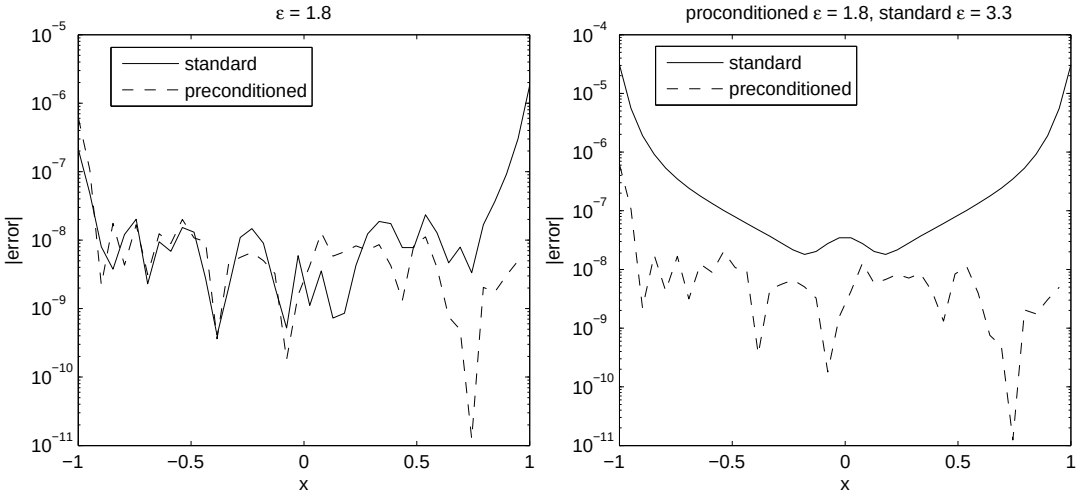

Figure 2: Accuracy of approximating the first derivative. Left: Standard ($\varepsilon = 1.8$) versus preconditioned ($\varepsilon = 1.8$) method. Right: Standard ($\varepsilon = 3.3$) versus preconditioned ($\varepsilon = 1.8$) method.

The boundary condition is enforced by setting the last row of $\bar{D}$ to zero.

We take $N = 40$ evenly spaced centers and use a shape parameter of $\varepsilon = 1.8$ which results in a system matrix with a condition number of $\kappa(B) = 8.4e16$. In figure 1, the scaled eigenvalues of the standard, D (also with the last row set to zero), and preconditioned $\bar{D}$ RBF differentiation matrices for problem (15) are shown with the stability region of an explicit fourth-order Runge-Kutta (RK4) method. The standard method is unstable using this value of the shape parameter. In a numerical study of RBF methods for time-dependent PDEs [16], it was shown that the method could be stabilized in many cases by increasing the shape parameter so that the condition number of system matrix has a condition number below a certain bound. In this problem, we find that shape parameters $\varepsilon \geq 3.3$ result in a stable method. With $\varepsilon = 3.3$, the condition number of the system matrix of the standard method is $\kappa(B) = \mathcal{O}(1e10)$.

The differentiation matrices D_x and $\bar{D}$ differ only in their elements on the main diagonal. The diagonal elements of the two differentiation matrices are plotted in figure 3. The diagonal elements are relatively large in rows corresponding to centers close to the boundaries when compared to diagonal elements in rows corresponding to interior points. Adding the diagonal cor-

rect term shown in the right image of figure 3 reduces the size of the diagonal entries in rows near where the boundary condition is applied. In this example the maximum diagonal element is 32.7 for D_x and is 12.8 for $\bar{D}$.

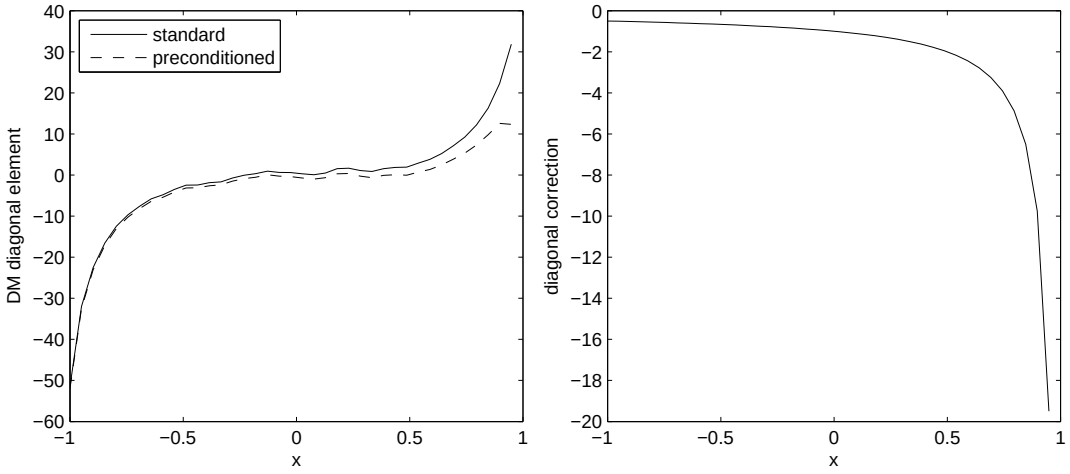

Figure 3: Left: Diagonal elements of the differentiation matrices D_x and $\bar{D}$ for problem (15). Right: The correction term $\frac{1}{x-1}$ that is added to the diagonal of D_x to get $\bar{D}$.

Before solving the PDE, we compare the accuracy of approximating a single first derivative by the standard and preconditioned methods. The function being differentiated is the exact solution to problem (15) at time $t = 0.4$. Figure 4 illustrates the results from a shape parameter of $\varepsilon = 1.8$ being used in both methods. The preconditioned method is more accurate near $x = 1$ where the boundary condition is to be applied and the two methods are of similar accuracy away from $x = 1$. In the right image, the smallest shape parameter is used for which each method is stable for time integration. The preconditioned method is much more accurate throughout the entire interval due to its ability to use a smaller shape parameter.

Both methods are used to advance the PDE solution to $t = 1$. The preconditioned method using $\varepsilon = 1.8$ has a maximum error of 2.4e-7 while the standard method with $\varepsilon = 3.3$ has a maximum error of 1.1e-5. The pointwise errors are plotted in figure 4. The preconditioned PDE solution is more accurate over the entire domain.

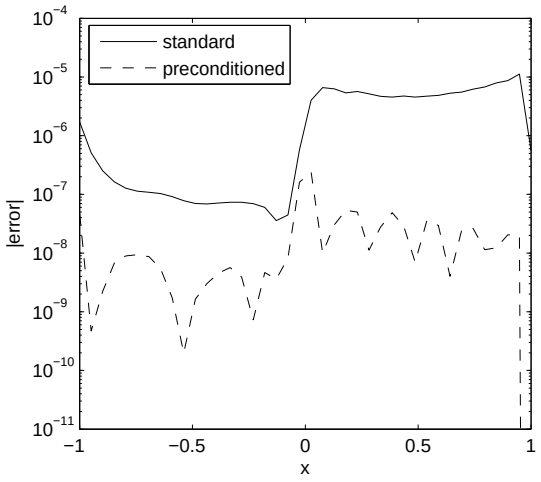

Figure 4: Standard and preconditioned RBF method errors at time $t = 1$ for the advection equation (15) with a zero Dirichlet boundary condition.

4.2 Advection with time-dependent BC

Next we consider the advection equation (15) where the initial condition and time-dependent boundary condition $u(1,t) = g(t)$ are taken from the exact solution $u(x,t) = \sin(2\pi(t + x + 1))$. The equation is preconditioned by assuming u is of the form

$$u(x,t) = \tau(x)w(x,t) + xg(t) \tag{19}$$

were $\tau(x) = 1 - x$. Then w is defined as

$$w(x,t) = \frac{u(x,t) - xg(t)}{\tau(x)} \tag{20}$$

at all x except for $x = 1$ where (20) is singular. At $x = 1$, L'Hospital's rule is applied to find that

$$w(1,t) = u_x(1,t) - g(t) \tag{21}$$

which is the boundary condition for the PDE in w. The derivative u_x is found by differentiating equation (19),

$$u_x = (x-1)D_1 w + w + g(t). \tag{22}$$

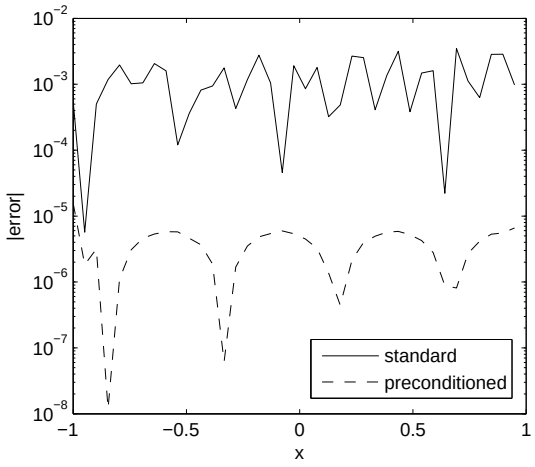

Figure 5: Standard versus preconditioned methods for the advection equation (15) with time-dependent boundary condition $u(1,t) = \sin(2\pi(t+2))$ at $t = 10$.

The standard RBF method is stable with $\varepsilon = 2.7$, $\kappa(B) = 1.5e12$. The problem is advanced to time $t = 10$ with $\Delta t = 0.001$ at witch time a maximum error of 3.5e-3 is recorded. The preconditioned method remains stable if the shape parameter is lowered to $\varepsilon = 2.0$ for which $\kappa(B) = 3.0e15$. At $t = 10$ the preconditioned method has a maximum error of 1.5e-5. The pointwise errors of the two methods are shown in figure 5. Throughout most of the domain the preconditioned method is about four decimal places more accurate.

4.3 Advection-Diffusion equation

Advection-Diffusion problems which are dominated by the advection term also are challenged by eigenvalue stability. For example, consider the advection diffusion equation

$$u_t + u_x = \nu u_{xx} \tag{23}$$

with Dirichlet boundary conditions at $x = -1$ and $x = 1$. To accommodate Dirichlet boundary conditions at both boundary points we assume the solution is of the form (10) and take the weight function to be $\tau(x) = 1 - x^2$.

To examine the stability properties of the preconditioned method we look

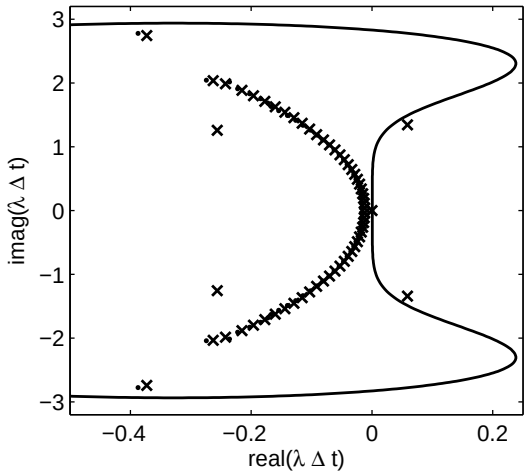

Figure 6: Advection-Diffusion problem eigenvalues. Standard x's and preconditioned ·'s.

at the PDE in the w variable

$$w_t = \nu w_{xx} + \left(\frac{2\nu\tau_x}{\tau} - 1\right) w_x + \left(\frac{\nu\tau_{xx} - \tau_x}{\tau}\right) w - \frac{1}{\tau}\left[u(-1,t) - u(1,t)\right] \quad (24)$$

with boundary conditions given by equation (12). The standard RBF method has the differentiation matrix

$$D = \mathrm{diag}(\nu) D_{xx} - D_x \quad (25)$$

while the preconditioned method has the differentiation matrix

$$\bar{D} = \mathrm{diag}(\nu) D_{xx} + \mathrm{diag}\left(\frac{2\nu\tau_x}{\tau} - 1\right) D_x + \mathrm{diag}\left(\frac{\nu\tau_{xx} - \tau_x}{\tau}\right). \quad (26)$$

We take $\nu = 0.0015$, zero Dirichlet boundary conditions, $N = 60$, and $\varepsilon = 2.8$. The eigenvalues of (25) and (26) scaled by $\Delta t = 0.025$ and the portion of the RK4 stability region near the imaginary axis are shown in figure 6. The condition number of the system matrix is $\kappa(B) = 1.1e17$. The standard method can be stabilized by increasing the shape parameter to $\varepsilon = 3.1$ for which the system matrix has condition number $\kappa(B) = 4.0e15$.

To compare the accuracy of the methods we take the initial condition to be

$$u(x,0) = \begin{cases} 1 & \text{when } x = -1 \\ 0 & \text{otherwise.} \end{cases}$$

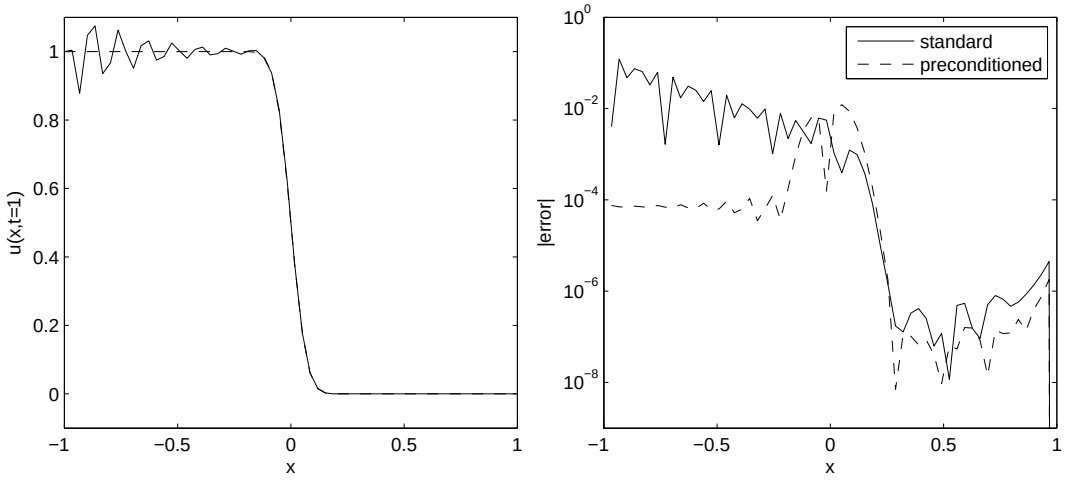

Figure 7: Left: Oscillatory standard RBF method solution of equation (23). Right: Standard versus preconditioned method errors at $t = 0.5$ for the advection-diffusion equation (23).

and Dirichlet boundary conditions from the exact solution

$$u(x,t) = \frac{1}{2}\left[\text{erfc}(\omega_1) + \exp\left(\frac{x}{\nu}\right)\text{erfc}(\omega_2)\right]$$

where $\omega_1 = (x - t + 1)/(2\sqrt{\nu t})$ and $\omega_2 = (x + t + 1)/(2\sqrt{\nu t})$. The solution is advanced to time $t = 0.5$ with $\Delta t = 0.001$. The standard method with $\varepsilon = 3.1$ has a maximum error of 1.2e-1 and a RMS error of 2.5e-2. The preconditioned method with $\varepsilon = 2.8$ has a maximum error of 1.2e-2 and a RMS error of 2.6e-3. The standard RBF solution featuring oscillations that are emanating from the boundary and the exact solution at $t = 0.5$ are shown in the left image of figure 7. The solution from the preconditioned method is free of oscillations as can be seen in the error plot in the right image of the figure. The non-oscillatory behavior of the preconditioned solution my be connected to the fact, as discussed in section 3, that the preconditioned method exactly differentiates constants while the standard method does not. The solution of the PDE is not constant, but has two regions were the solution is flat.

For this problem, the boundary conditions as given by equation (12) are

$$w(-1,t) = \frac{u_x(-1,t) - \frac{1}{2}[u(1,t) - u(-1,t)]}{\tau_x(-1)} \tag{27}$$

and

$$w(1,t) = \frac{u_x(1,t) - \frac{1}{2}[u(1,t) - u(-1,t)]}{\tau_x(1)}. \tag{28}$$

We note that u_x is calculated by

$$u_x(-1,t) = \sum_{j=1}^{N} (D_x)_{1,j} \cdot u(x_j) \tag{29}$$

and

$$u_x(1,t) = \sum_{j=1}^{N} (D_x)_{N,j} \cdot u(x_j). \tag{30}$$

4.4 Wave equation

Next we consider the wave equation

$$u_{tt} = u_{xx} \tag{31}$$

on $[-1,1]$ and boundary conditions $u(-1) = 0$ and $u(1) = 0$. The initial condition is $u(x,0) = \sin(\pi x) + 0.5\sin(3\pi x)$.

To advance the wave equation in time we use Störmer's second order method for $u_{tt} = F(u)$ which is

$$u^{n+1} = 2u^n - u^{n-1} + \Delta t^2 F(u^n). \tag{32}$$

In the $\Delta t^2 \lambda$ plane, the stability interval of Störmer's method is the interval $[-4, 0]$.

The solution is assumed to be of the form (10) and we take the weight function to be $\tau(x) = 1 - x^2$. To examine the stability properties of the preconditioned method we look at the PDE in w

$$w_{tt} = w_{xx} + \left(\frac{2\tau_x}{\tau}\right) w_x + \left(\frac{\tau_{xx}}{\tau}\right) w \tag{33}$$

and boundary conditions

$$w(-1,t) = u_x(-1,t)/\tau_x(-1) \tag{34}$$

and

$$w(1,t) = u_x(1,t)/\tau_x(1). \tag{35}$$

The standard RBF method has the differentiation matrix

$$D = D_{xx} \tag{36}$$

while the preconditioned method has the differentiation matrix

$$\bar{D} = D_{xx} + \text{diag}\left(\frac{2\tau_x}{\tau}\right) D_x + \text{diag}\left(\frac{\tau_{xx}}{\tau}\right). \tag{37}$$

With $N = 40$ evenly spaced centers, the smallest value of the shape parameter for which the standard method is stable is $\varepsilon = 2.7$ while the preconditioned method may use a shape parameter as small as 2.1. The scaled eigenvalues of D and $\bar{D}$ with $\varepsilon = 2.1$ are shown in figure 8.

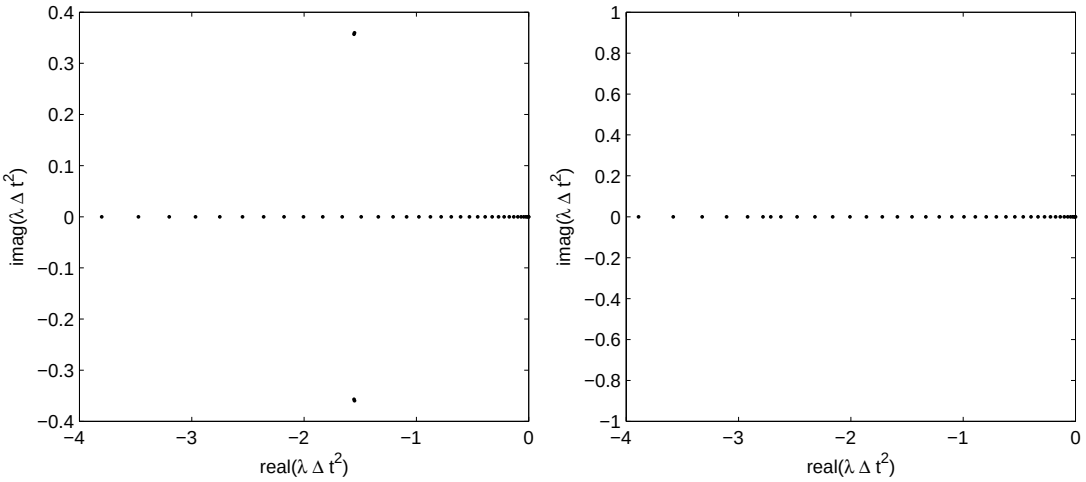

Figure 8: Eigenvalues scaled by Δt^2, $\Delta t = 0.033$, $N = 40$ and $\varepsilon = 2.1$. Left: standard method, maximum imaginary part of differentiation matrix eigenvalues is 330.3. Right: preconditioned method, all eigenvalues are real and negative.

While improving stability, the preconditioned method is only marginally more accurate than the standard method for the wave equation. This is most likely due to the formula (14) for the preconditioned second derivative. While multiplication by τ does mitigate the boundary errors in w_{xx}, the boundary errors in w_x are not lessened by multiplication by $\tau_x = -2x$. We have tried other weights, but were unable to find any that resulted in further

improvements in accuracy. Another way to calculate the preconditioned second derivative is to use equation (13) recursively. In examples, this approach was not significantly more accurate.

4.5 2d Advection equation

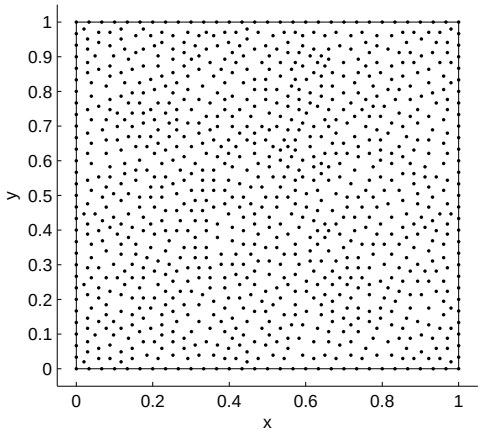

Figure 9: $N = 1000$ scattered centers used in problem 38.

Finally, we consider the 2d advection equation

$$u_t + u_x + u_y = 0 \tag{38}$$

on the unit square $[0,1]\times[0,1]$ with Dirichlet boundary conditions, $u(0,y,t) = 0$ and $u(x,0,t) = 0$. The scattered centers in figure 9 are distributed by the algorithm in [13] which locates the centers in a "near optimal" manner such that the domain is well covered but no two centers are close enough to adversely affect conditioning. The initial condition is $u(x,y,0) = \exp(-150((x-0.5)^2 + (y-0.5)^2))$ and the weight $\tau(x,y) = xy$ is used in the preconditioned method.

In the variable w, the preconditioned PDE is

$$w_t + w_x + w_y + \left(\frac{\tau_x + \tau_y}{\tau}\right) w = 0 \tag{39}$$

with boundary conditions $w(x,0,t) = 0$ and $w(0,y,t) = 0$. The differentiation matrix for (39) is

$$\bar{D} = D - \text{diag}\left(\frac{\tau_x + \tau_y}{\tau}\right) \tag{40}$$

where the differentiation matrix for the standard method is $D = -(H_x + H_y)B^{-1}$. The boundary conditions are enforced by setting the rows of D and $\bar{D}$ that coincide to boundary points to zero.

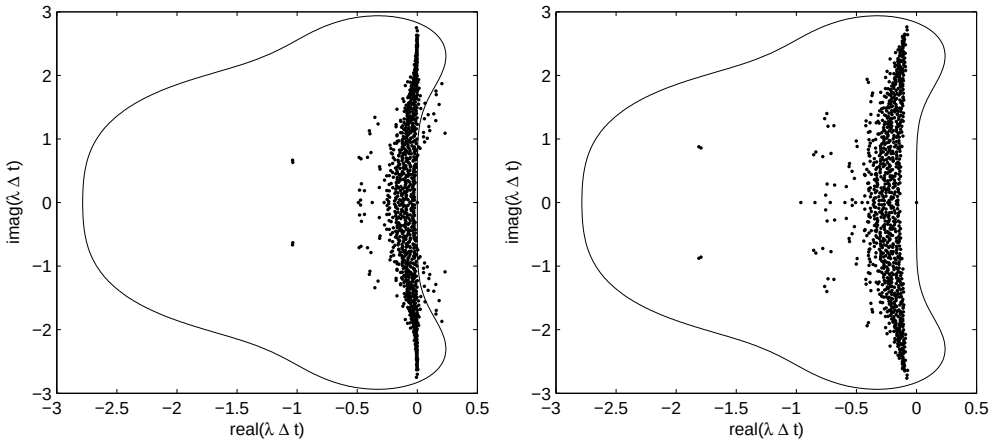

Figure 10: Equation (38). Standard and preconditioned RBF methods with $\varepsilon = 3.75$, $\kappa(B) = 4.3e17$ and $N = 1000$ centers. The eigenvalues are scaled by $\Delta t = 0.02$. Left: standard method with maximum real eigenvalue of 11.6. Right: preconditioned method with maximum real eigenvalue of zero.

For eigenvalue stability, the standard method can not use a shape parameter smaller than $\varepsilon = 6$ for which $\kappa(B) = 7.0e12$. The preconditioned method can use a shape parameter as small as $\varepsilon = 3.75$ which results in $\kappa(B) = 4.3e17$. The eigenvalues of the differentiation matrices of the two methods are shown in figure 10. The methods are advanced in time with $\Delta t = 0.001$ to time $t = 0.4$ at which time the initial concentration is beginning to leave the domain. The standard method maximum error is 1.0e-4 and the preconditioned method has a maximum error of 3.3e-5 (upper images of figure 11). The two methods are advanced further to $t = 10$ at which time the exact solution is zero. The standard and preconditioned method errors are respectively 5.3e-5 and 5.6e-24 (lower images of figure 11).

5 Chebyshev pseudospectral method eigenvalue stability

We remarked in the introduction that the idea of preconditioning derivatives [10, 2, 5] originated in connection with the Chebyshev pseudospectral (CPS) method [1, 4, 11]. The preconditioning technique was used only to enhance accuracy and its effect on eigenvalue stability was not considered. We consider the CPS method for the advection equation (15) with boundary condition $u(1,t) = 0$ with $N = 20$ Chebyshev-Gauss-Lobatto grid points $x_k = -\cos(k\pi/(N-1))$, $k = 0, 1, \ldots, N-1$. The eigenvalues of the standard and precondition CPS differentiation matrices are shown in figure 12. The effect of preconditioning on eigenvalue stability of the CPS method is that it shifts some of the eigenvalues so that they have large negative real parts. Thus, if explicit time integration is used, the preconditioned CPS method will have to use a much smaller time step than the standard method.

6 Conclusions

PDEs that are discretized by the RBF collocation method often have differentiation matrices with eigenvalues that have large positive real parts and therefore can not be stably advanced in time. The preconditioning technique that we have described reduces errors near points where boundary conditions are applied and enhances the stability properties of the RBF collocation method for time-dependent PDEs. The application of the preconditioning techniques requires the specification of a weight function which is zero at points where boundary conditions are applied and then increases in value towards the interior of the domain. We have only considered uniform or near uniform center placement in order to illustrate the effectiveness of the preconditioning technique. However, preconditioning can be used in conjunction with other techniques that may reduce boundary region errors or improve stability, such as center clustering.

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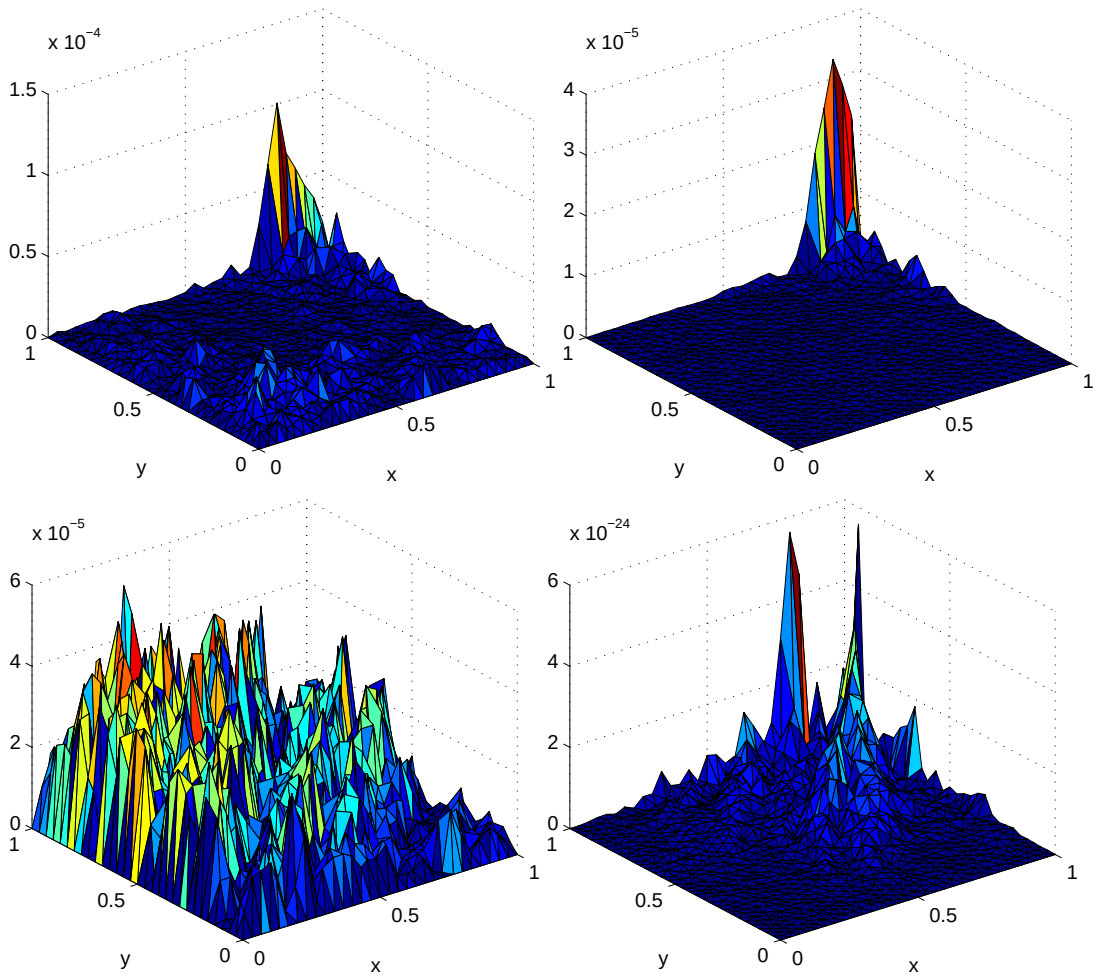

Figure 11: Error plots for problem (38). Upper left: Standard method, $t = 0.4$. Upper right: Preconditioned method, $t = 0.4$. Lower left: Standard, $t = 10$. Lower right: Preconditioned, $t = 10$.

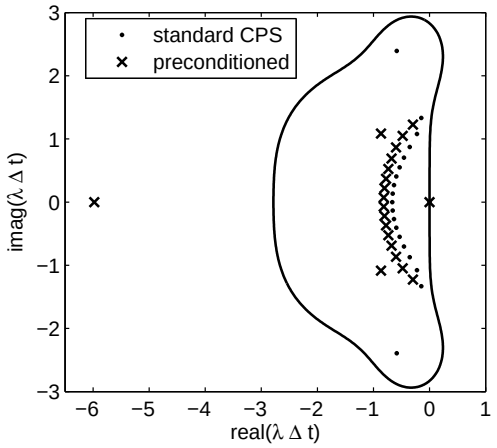

Figure 12: Eigenvalues scaled by $\Delta t = 0.075$ of the standard and preconditioned Chebyshev pseudospectral method for problem (15).