

Postprocessing High Order Approximation Methods

Scott A. Sarra

scott@ScottSarra.org

<http://www.ScottSarra.org>

Department of Mathematics

Marshall University

Talk Outline

- Global High Order Approximation Methods
 - Polynomial (Pseudospectral)
 - Radial Basis Function (RBF)
- The Gibbs Phenomenon
- Postprocessing Methods
- The Matlab Postprocessing Toolbox* (MPT)

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Global HOAMS

$$\mathcal{I}_N f(x) = \sum_k a_k \phi_k(x)$$

- $\mathcal{I}_N f(x_i) = f(x_i)$ at $N + 1$ interpolation sites x_i
- *Interpolation* - $f(x)$ is a known function (at least at the interpolation sites)
- *collocation* and *pseudospectral* - global interpolation based methods for solving differential equations for an unknown function $f(x)$

Basis Functions

- Polynomial - structured grids/domains
 - Periodic - Fourier (Trigonometric)

$$\phi_k(x) = e^{ik\pi x}, \quad x_k = -1 + \frac{2}{N}$$

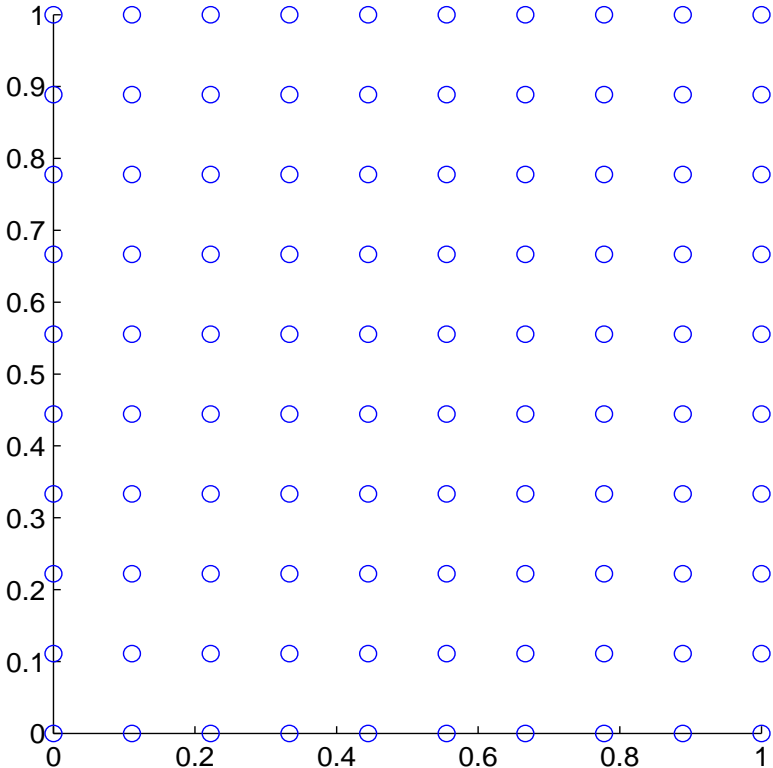
- Non-periodic - Chebyshev (or Legendre)

$$\phi_k(x) = \cos(k \arccos(x)), \quad x_k = -\cos(k\pi/N)$$

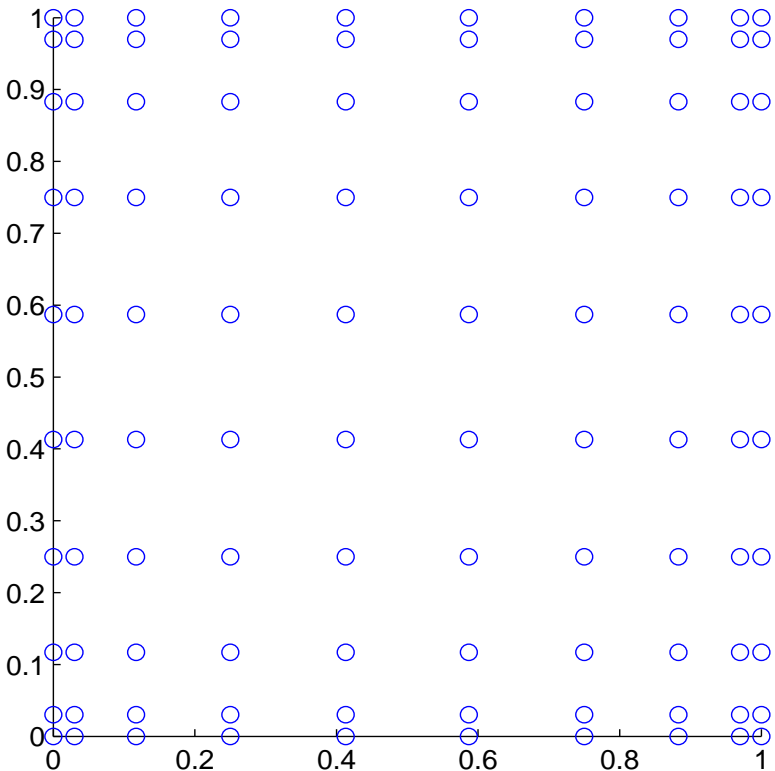
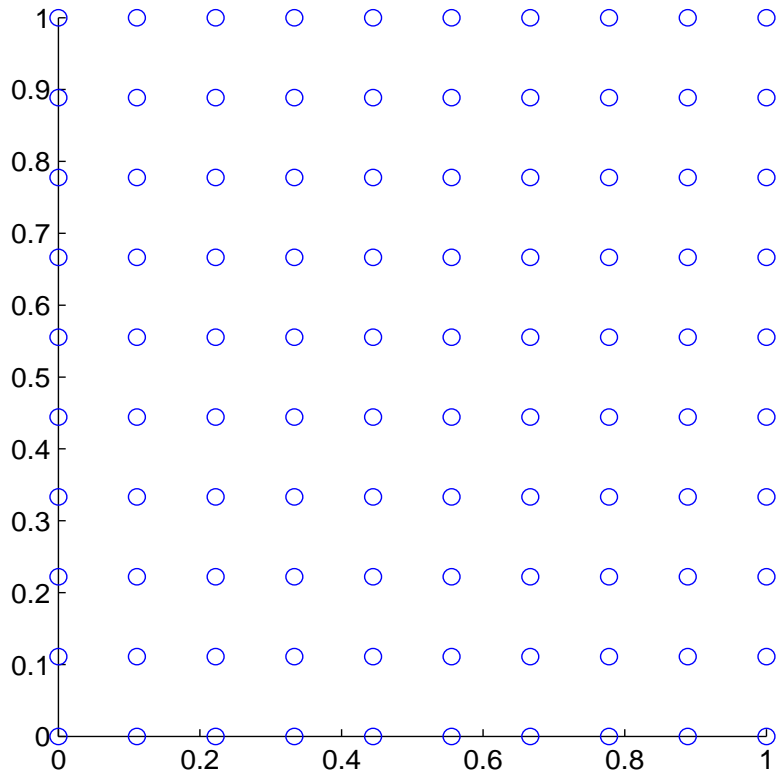
- RBFs: scattered centers in complex domains

$$\phi_k(r = \|x - x_k\|_2, \varepsilon) = \sqrt{1 + \varepsilon^2 r^2}$$

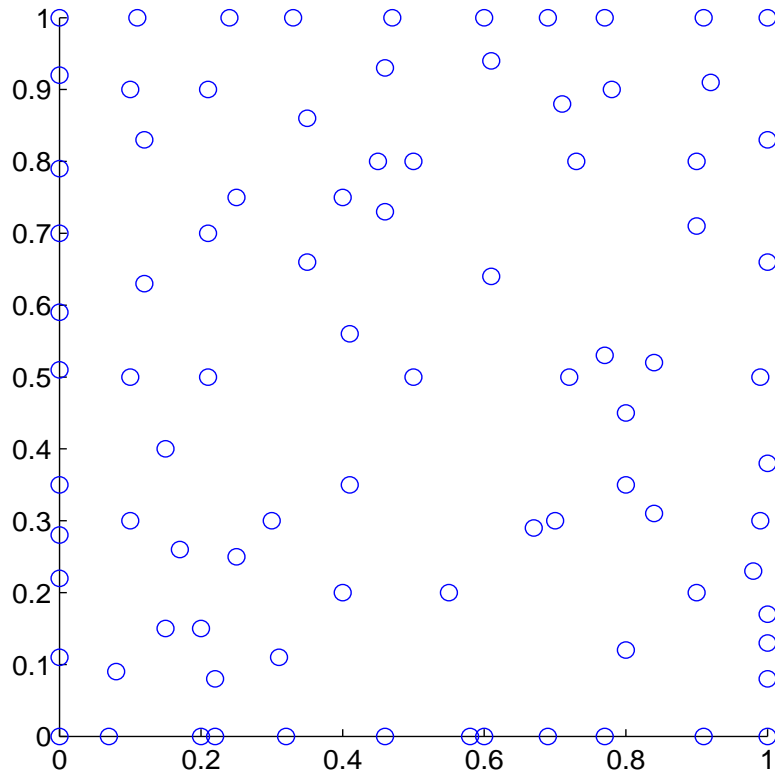
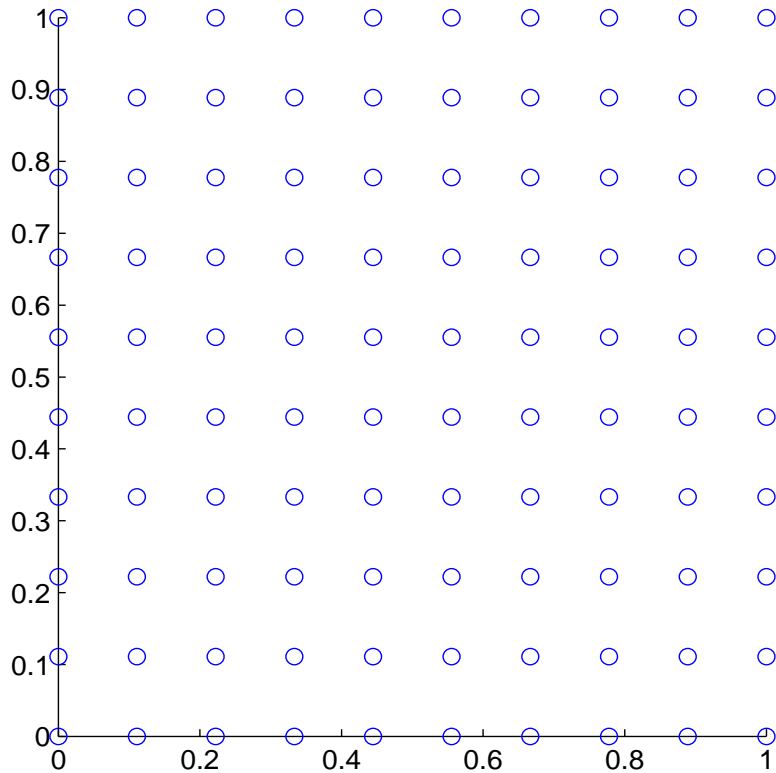
Grids/Domains



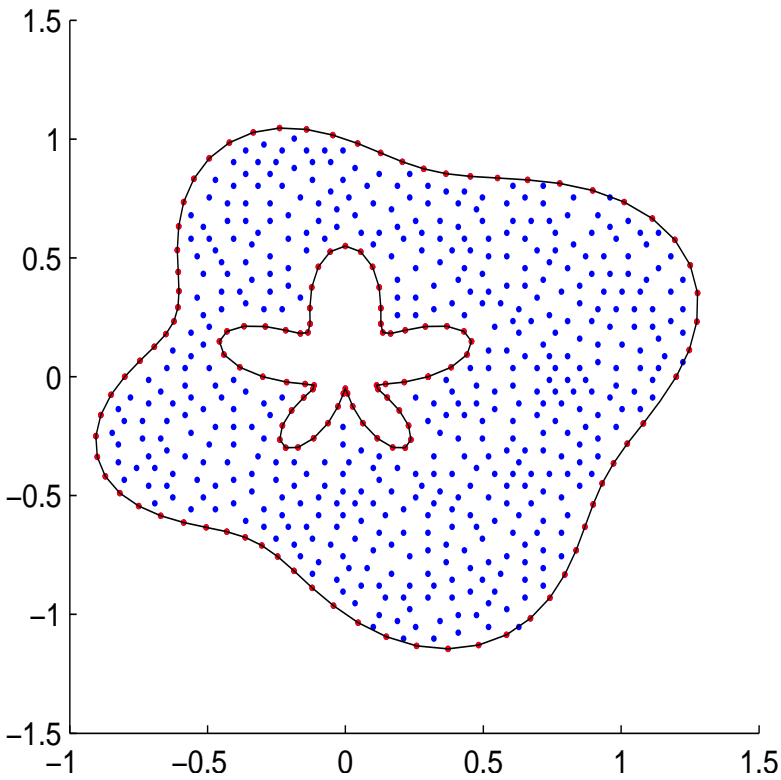
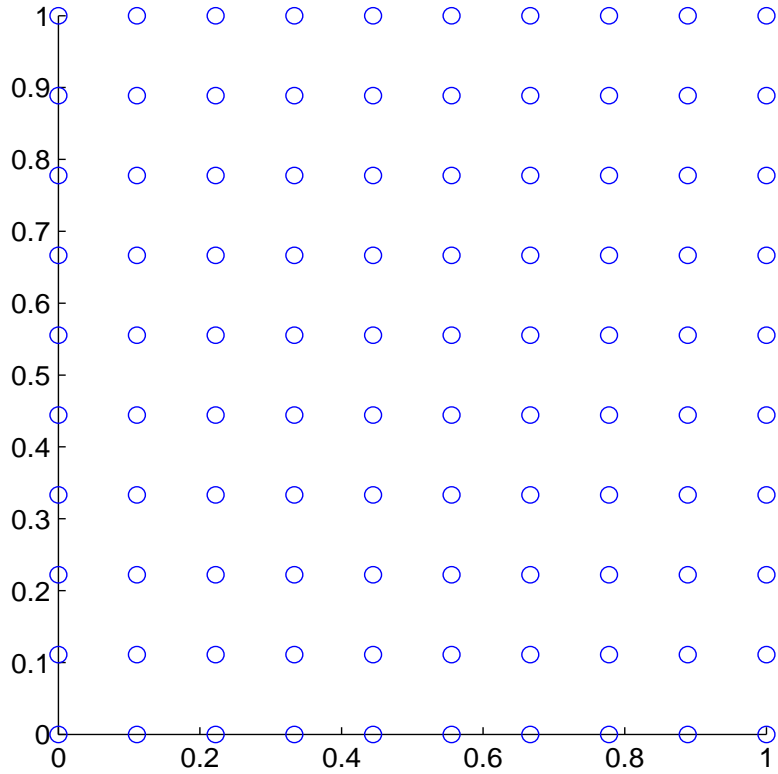
Grids/Domains



Grids/Domains



Grids/Domains



RBFs

- Simple, Grid/Geometry free, Spectrally accurate.
- Extend trivially to any number of spatial dimensions.
- Ill-conditioning, Efficient implementation, Stability for time-dependent PDE problems.
- “small” $\varepsilon \Rightarrow$ better accuracy, worse conditioning of the system matrix.
- Uncertainty relation: The error and the condition number cannot both be kept small
- \nexists a formula to specify the optimal value of ε .
- Computation with the “optimal” ε is often not possible with standard algorithms.
- In the limiting case $\varepsilon \rightarrow 0$ the RBF interpolant is a polynomial interpolant in 1d.
- RBF methods - generalized spectral methods

Spectral Accuracy

- Spectral
 - $\mathcal{O}(N^{-m})$ (for every m) - infinitely differentiable
 - $\mathcal{O}(c^N)$, $0 < c < 1$ - analytic
- RBF

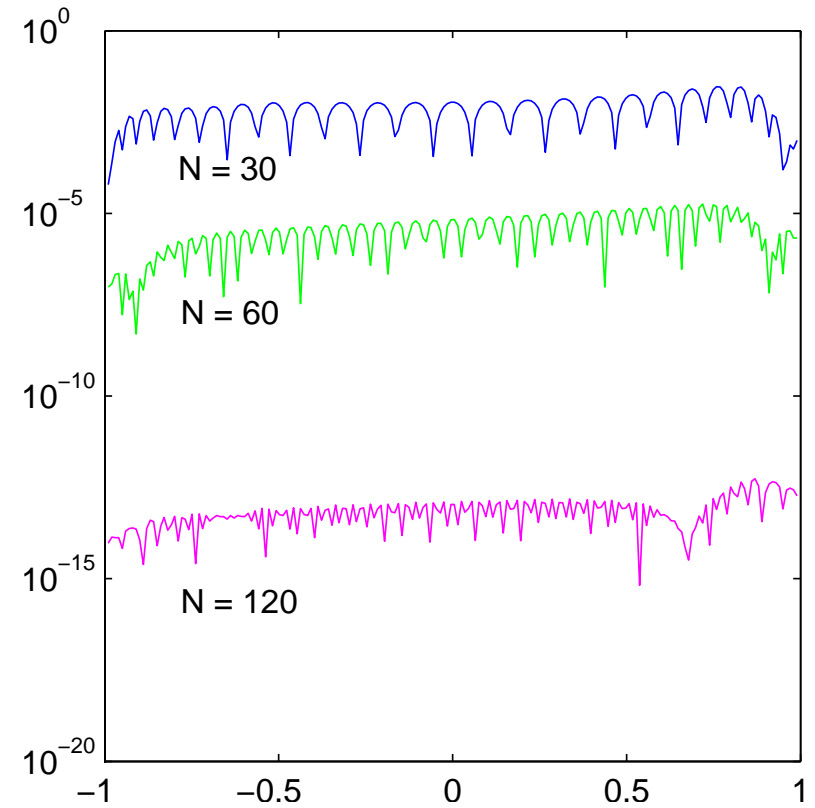
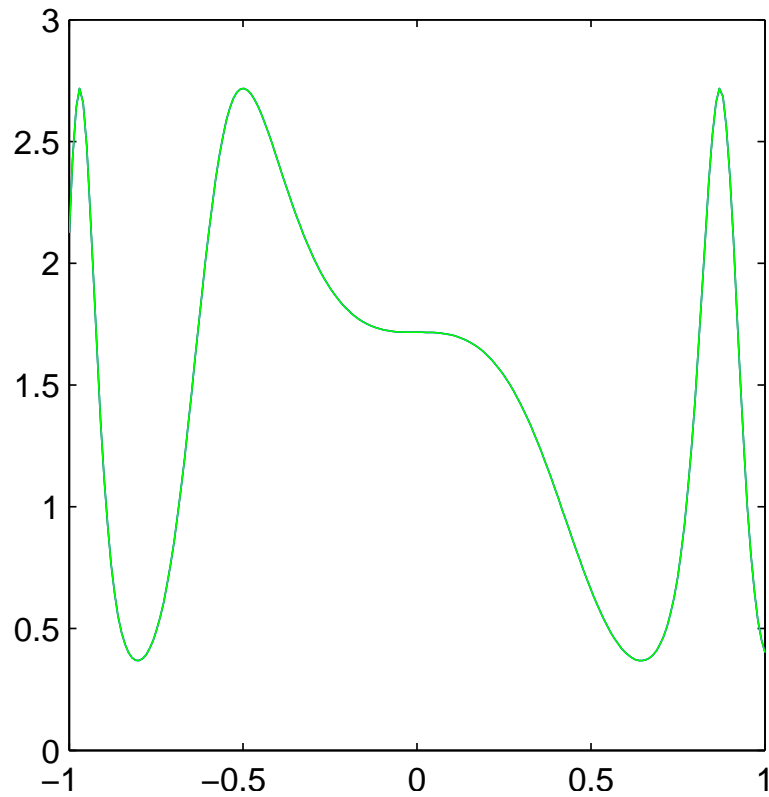
$$\mathcal{O}\left(e^{\frac{-K(\varepsilon)}{h}}\right)$$

where the *fill distance*

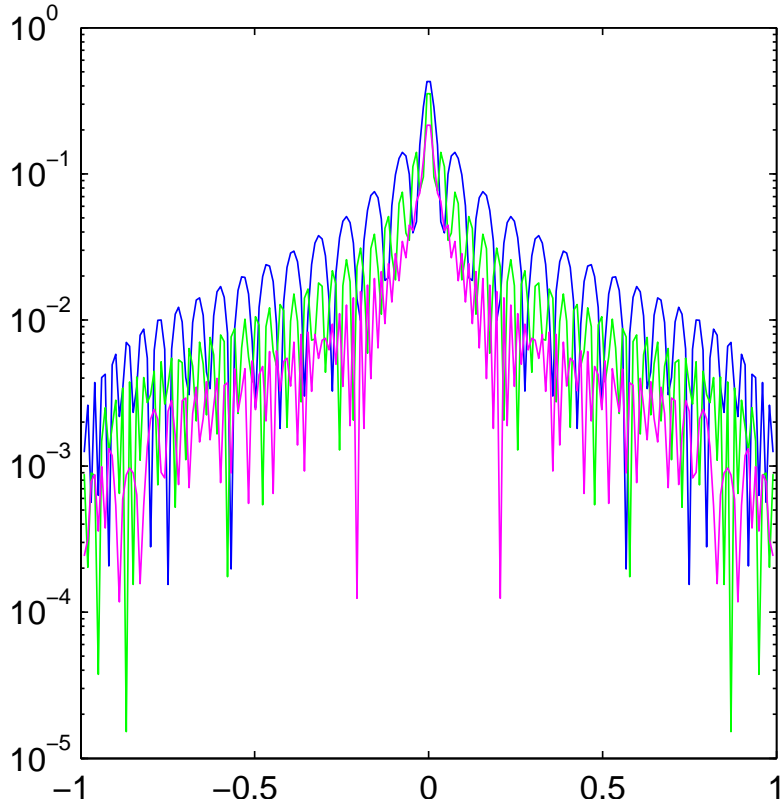
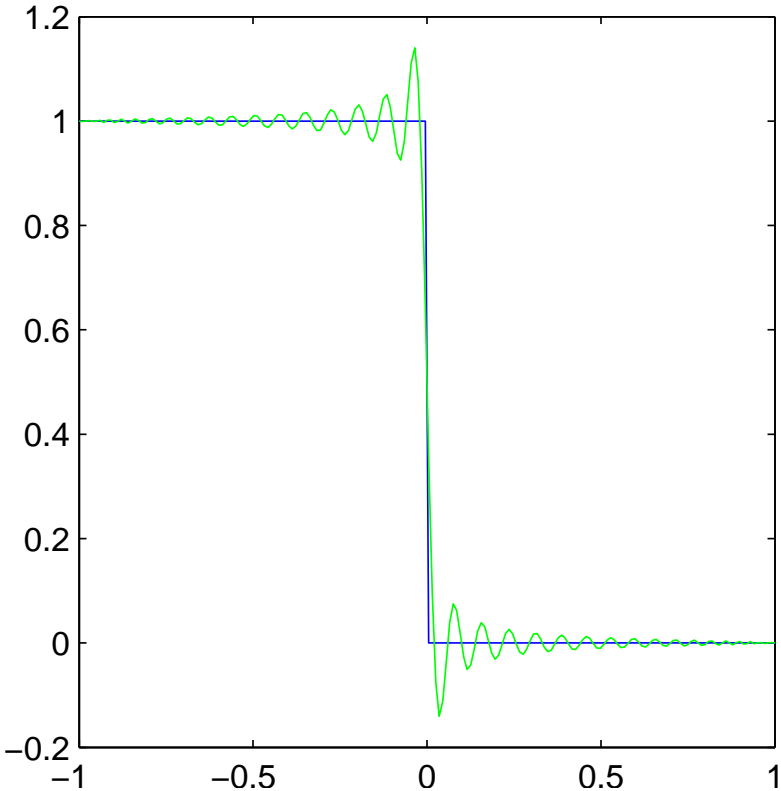
$$(1) \quad h_{\Xi} = \sup_{\xi \in \Xi} \min_{\xi_j \in \Xi} \|\xi - \xi_j\|.$$

and $K(\varepsilon)$ is a constant that depends on the value of the shape parameter.

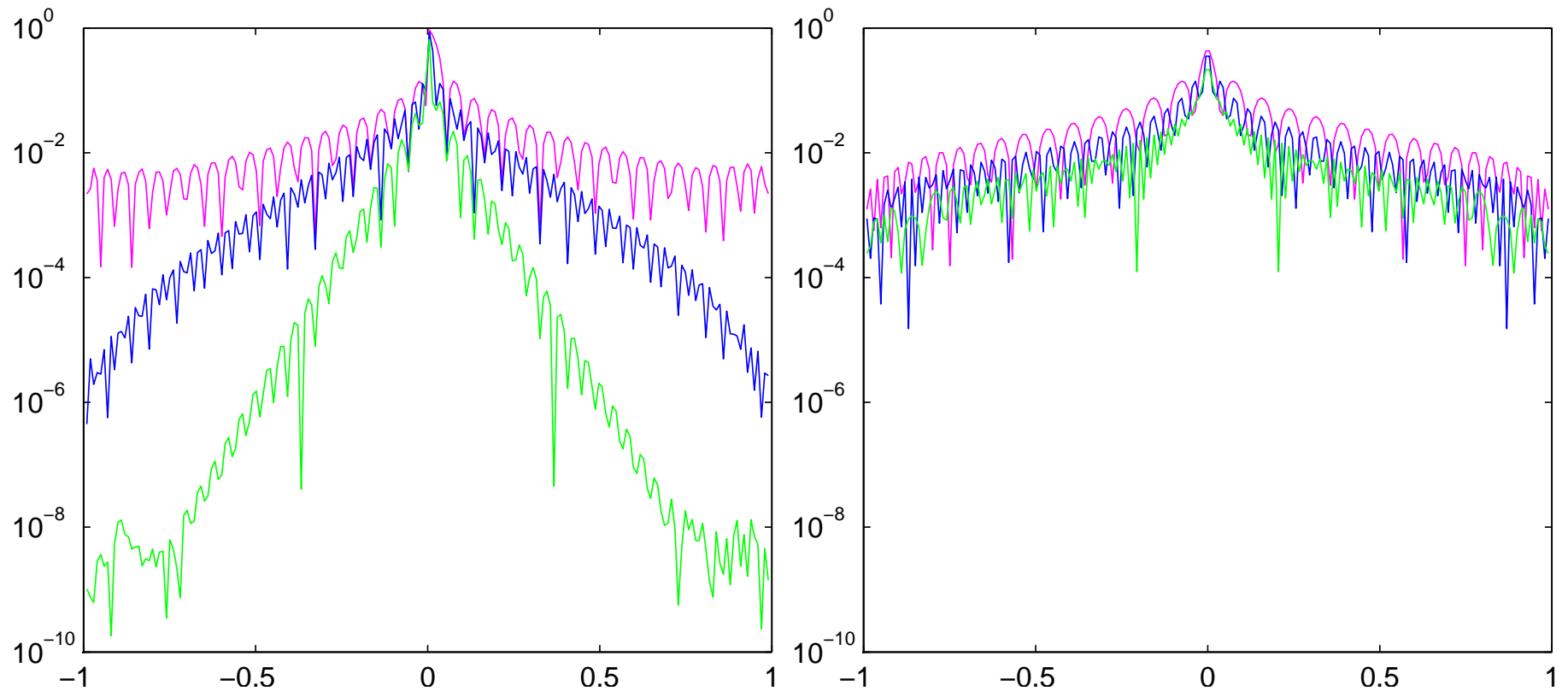
Spectral Accuracy



Gibbs Phenomenon



RBF vs. Pseudospectral Gibbs



From top to bottom, $N = 39, 79, 159$. Left: MQ RBF. Right: Chebyshev.

Pseudospectral events

- Fourier (1822)
- Wilbraham (1848)
- Gibbs (1898)
- Fejer (1900) - 1st order spectral filter
- Lanczos (1938) - application to ODEs
- Cooley and Tukey (1965) - FFT
- Kreiss and Oliger (1972) - Fourier spectral collocation
- Orzag (1972) - coined term pseudospectral
- Gotlieb and Orzag (1977) - theory

Pseudospectral events

- Gottlieb and Tadmor - (1984) physical space filters
- Vandeven (1991) - spectral filters
- Gottlieb, et. al (1992) - Gegenbauer reprojection
- Gelb and Tadmor (1999) - edge detection
- Shizgal and Jung (2003) - inverse reprojection
- Gelb and Tanner (2006) - Freud reprojection
- Sarra (2006) - DTV filtering

RBF events

- Hardy (1971) - MQ RBF for scattered interpolation
- Franke (1979) - survey of scattered methods
- Micchelli (1986) - RBF system matrix invertible
- Kansa (1990) - RBF methods for PDEs
- Madych and Nelson (1992) - spectral convergence
- Driscoll and Fornberg (2002) - connection to Lagrange interpolant
- Sarra (2006) - DTV filtering

Postprocessing Methods

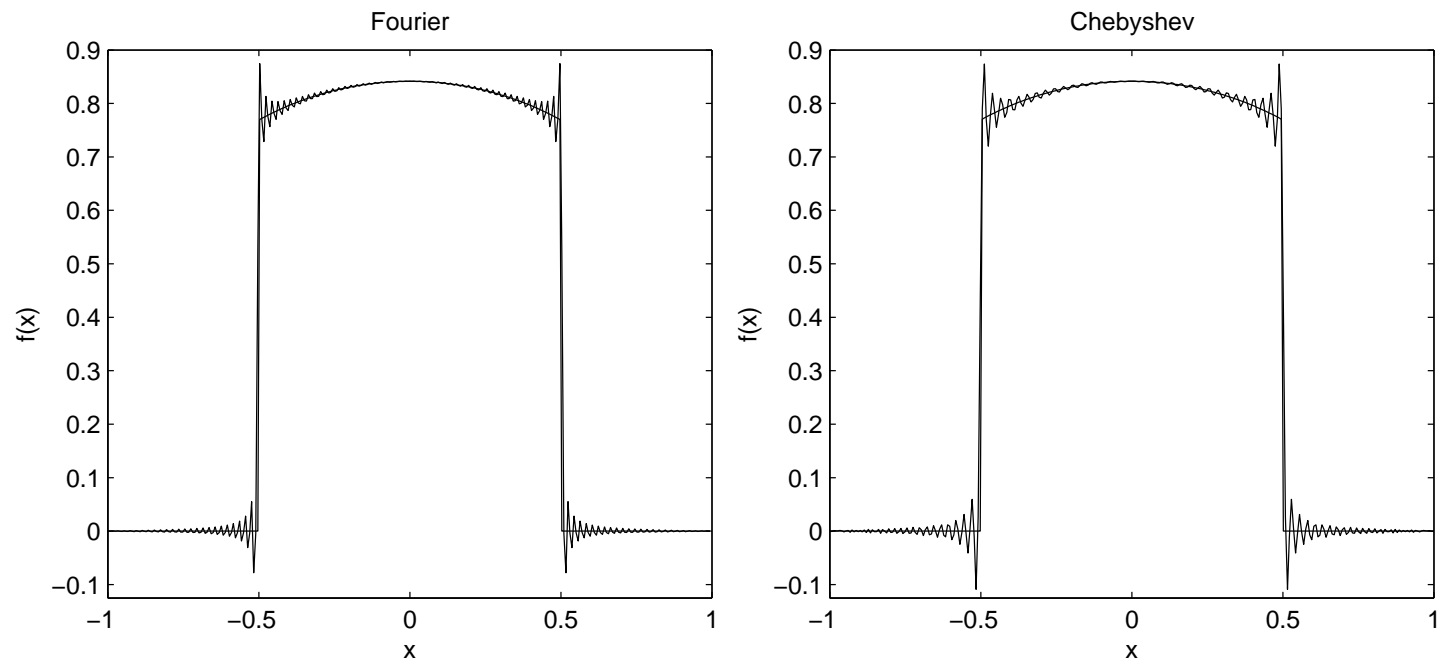
- Spectral filtering
- Rational Reconstruction (*)
- Reprojection Methods *
 - Gegenbauer *
 - Freud *
- Inverse Polynomial Reprojection *
- Digital Total Variation Filtering
- Hybrid Method

* Requires edge detection.

Matlab Postprocessing Toolbox

- collection of Matlab functions
- edge detection and postprocessing algorithms
- Fourier and Chebyshev
- one and two space dimensions
- applications, algorithm benchmarking, and educational purposes
- graphical user interface
- <http://www.scottsarra.org/mpt/mpt.html>

Test Function



$$f(x) = \chi_{[-0.5, 0.5]} * \sin[\cos(x)]$$

Spectral Filter

$$\mathcal{F}_N f(x) = \sum_k \sigma(k/N) a_k \phi_k(x)$$

- Exponential Filter

$$\sigma_1(\omega) = e^{-(\ln \varepsilon_m) \omega^\rho}$$

- Erfc-Log filter

$$\sigma_2(\omega) = \frac{1}{2} \operatorname{erfc} \left(2\sqrt{\rho} [|\omega| - 1/2] \sqrt{\frac{-\ln(1 - 4[|\omega| - 1/2]^2)}{4[|\omega| - 1/2]^2}} \right)$$

- Vandeven filter

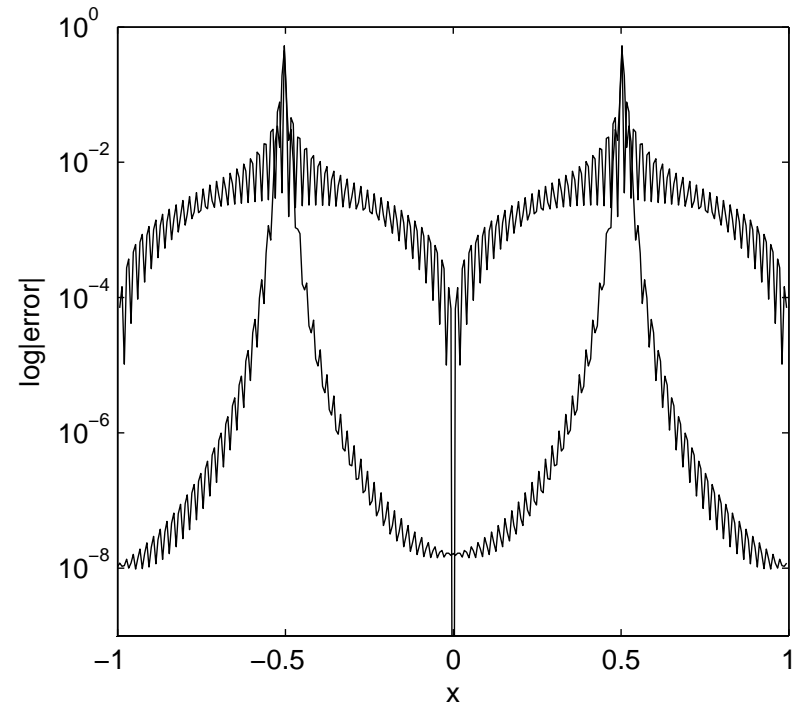
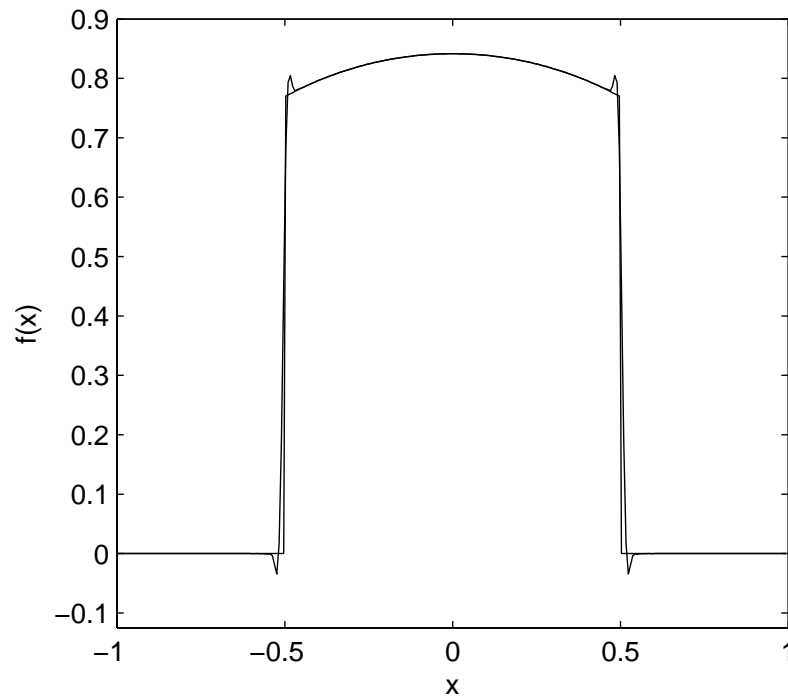
$$\sigma_3(\omega) = 1 - \frac{(2\rho - 1)!}{(p - 1)!} \int_0^{|\omega|} t^{\rho-1} (1 - t)^{\rho-1} dt$$

Spectral Filter

$$|f(x) - \mathcal{F}_N(x)| \leq \frac{K}{d(x)^{\rho-1} N^{\rho-1}}$$

- discontinuity at x_0 , $d(x) = x - x_0$
- ρ sufficiently large, $d(x)$ not too small, error goes to zero faster than any finite power of $1/N$, i.e. spectral accuracy is recovered.
- If $d(x) = \mathcal{O}(1/N)$ then the error estimate is $\mathcal{O}(1)$.
- extends easily to higher dimensions

Spectral Filter Example



Vandeven Filtered $\rho = 4$ Fourier approximation and error.

Rational Reconstruction

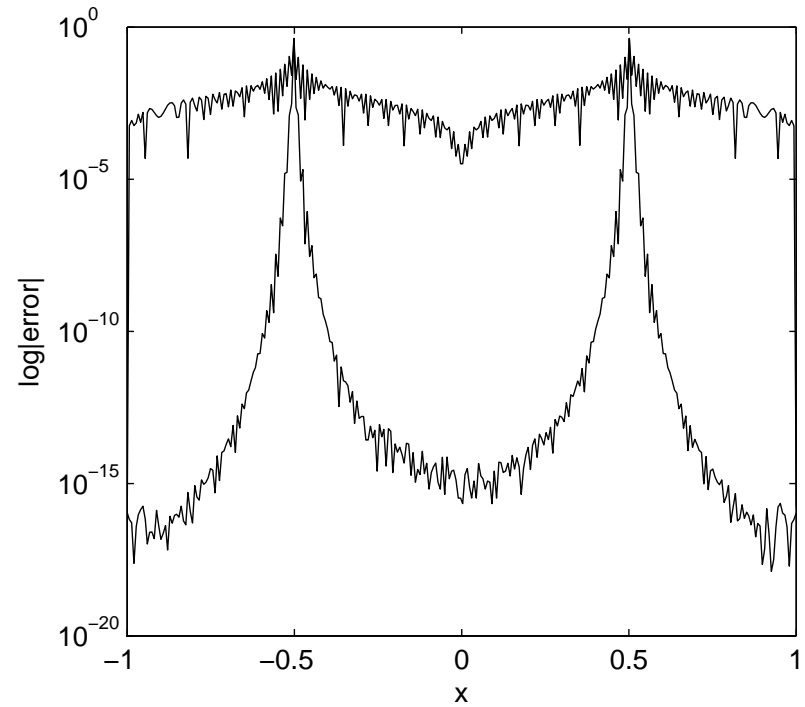
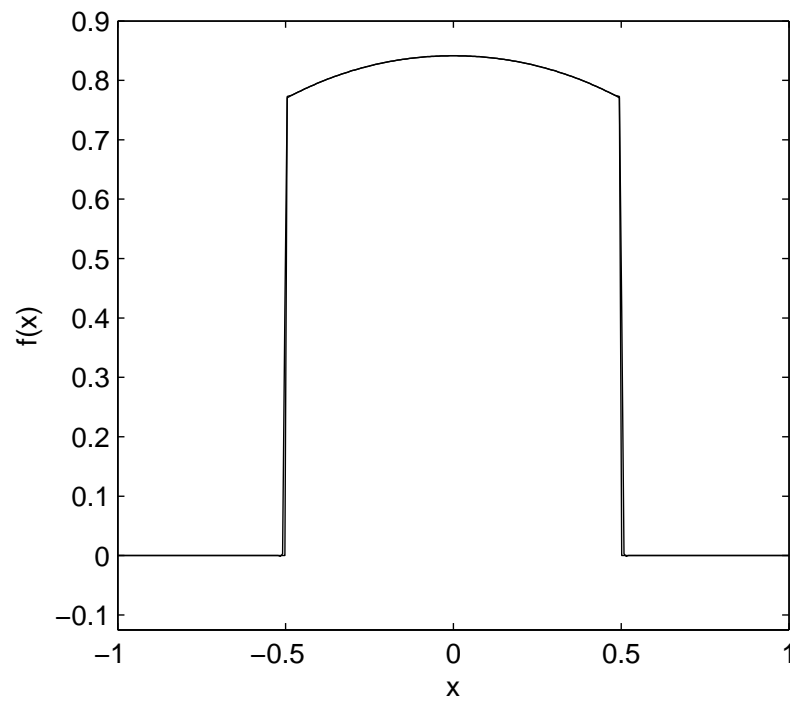
$$R_{K,M} = \frac{P_K}{Q_M} = \frac{\sum_{k=0}^K p_k \phi_k(x)}{\sum_{m=0}^M q_m \phi_m(x)}$$

- determined by imposing the orthogonality relations

$$\langle Qu - P, \phi \rangle = 0, \quad \forall \phi \in \mathbb{P}_N$$

- a (ill-conditioned) linear system must be solved
- may incorporate edge detection
- can be applied “slice by slice” in higher dimensions
- high numbered spectral coefficients may contain significant noise and should not be used
- lacks theoretical support - works mysteriously

Rational Reconstruction Example



Chebyshev Rational Reconstruction.

Reprojection Methods

In each of the i smooth subintervals $[a, b]$ the function is projected or reprojected as

$$f_P^i(x) = \sum_{\ell=0}^{m_i} g_\ell^i \Psi_\ell[\xi(x)]$$

- $\xi(x)$ takes $x \in [a, b]$ to $\xi \in [-1, 1]$ and $x(\xi)$ is the inverse map
- the reprojection basis $\Psi_\ell(x)$, orthogonal polynomials on $[-1, 1]$ wrt a weight $w(x)$
- weighted inner product

$$(\Psi_k(\xi), \Psi_\ell(\xi))_w = \int_{-1}^1 \Psi_k(\xi) \Psi_\ell(\xi) w(\xi) d\xi = \gamma_\ell \delta_{kl}$$

- The reprojection coefficients g_ℓ^i are evaluated via a Gaussian quadrature formula.
- γ_ℓ is a normalization factor

Reprojection Methods

Exact Reprojection Coefficients (projection)

$$g_\ell^i = \frac{1}{\gamma_\ell^\lambda} \int_1^{-1} w(\xi) \Psi_\ell^\lambda(\xi) f[x(\xi)] d\xi$$

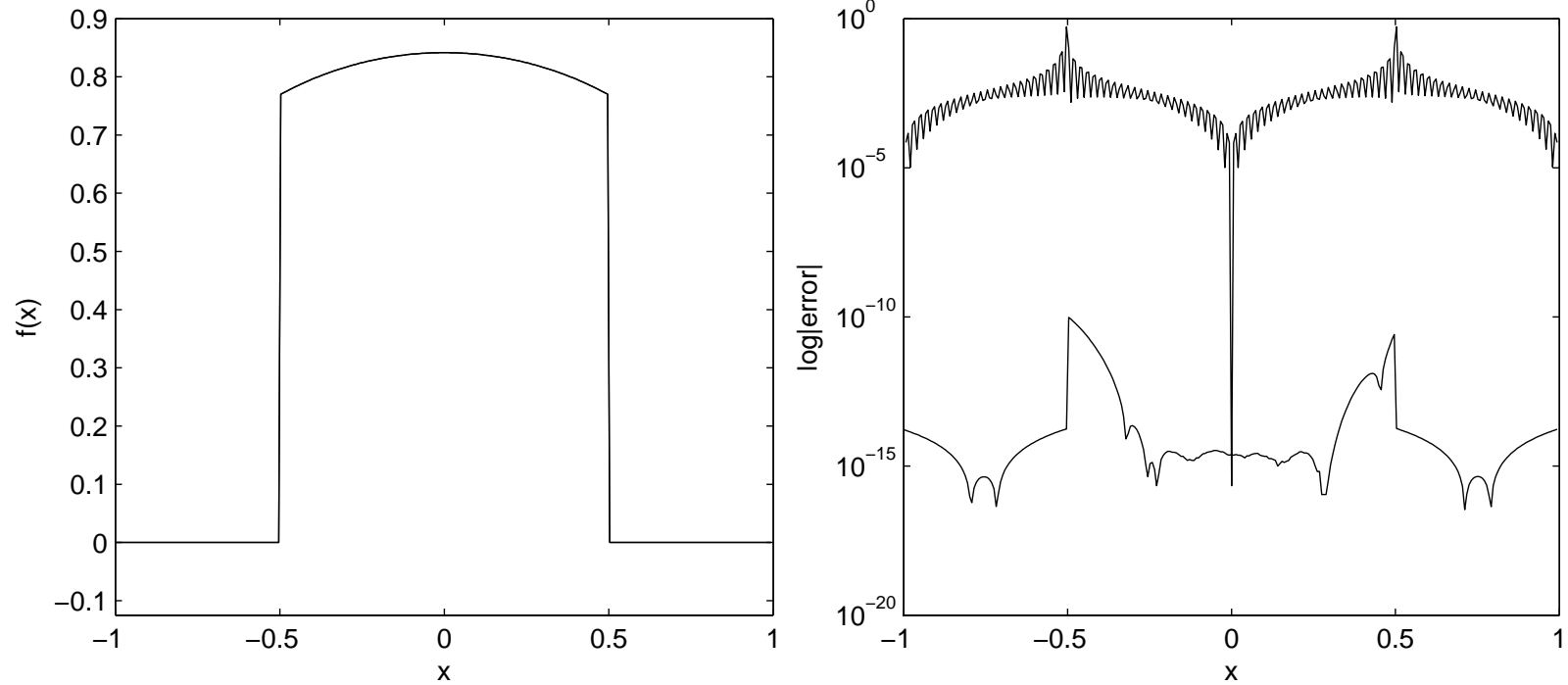
Approximate Reprojection Coefficients (reprojection)

$$\hat{g}_\ell^i = \frac{1}{\gamma_\ell^\lambda} \int_1^{-1} w(\xi) \Psi_\ell^\lambda(\xi) \mathcal{I}_N f[x(\xi)] d\xi$$

Gegenbauer Reprojection

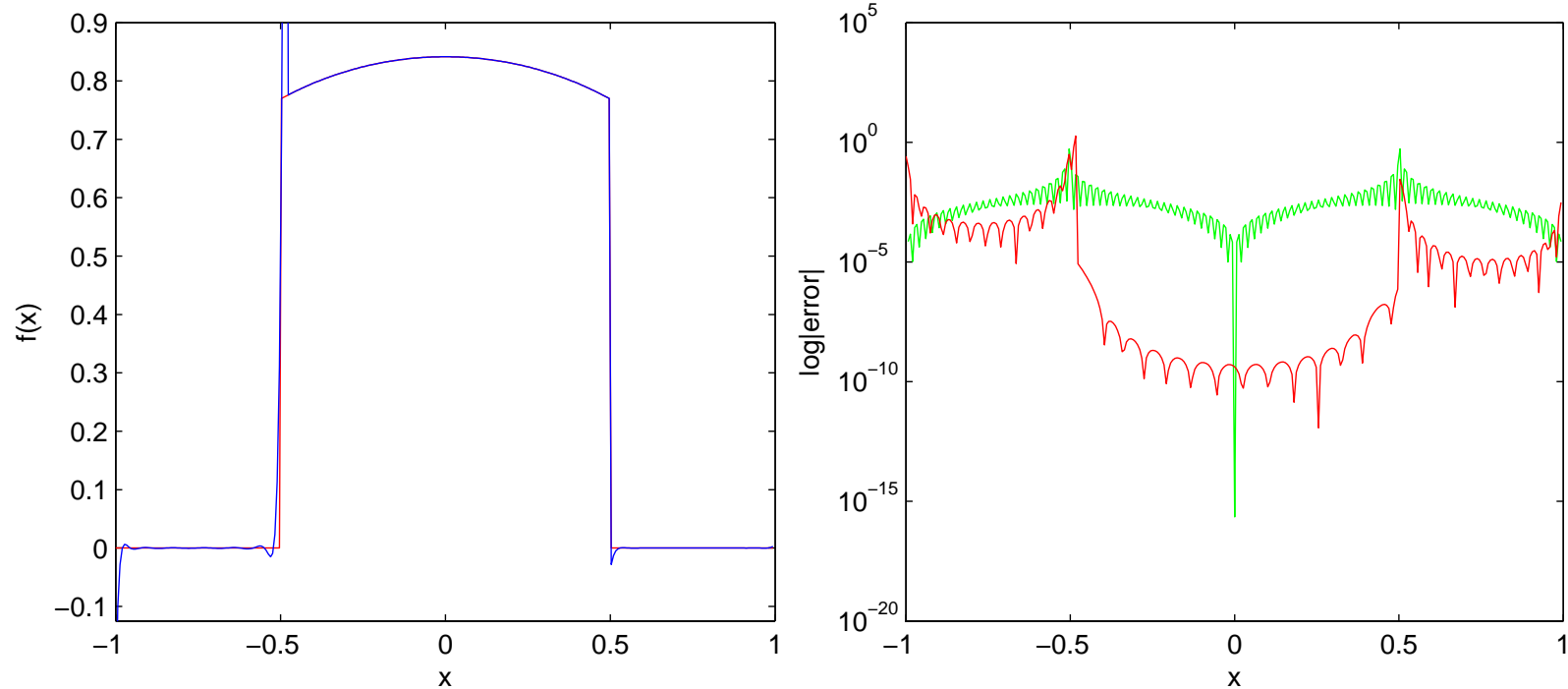
- Ψ_ℓ^λ Gegenbauer or Ultraspherical polynomials as the reprojection basis
- $w(x) = (1 - x^2)^{\lambda-1/2}$
- not known in closed form - calculated via a three term recurrence relationship
- accurate edge detection critical
- function dependent parameters λ and m
- noise

Gegenbauer Reprojection Example



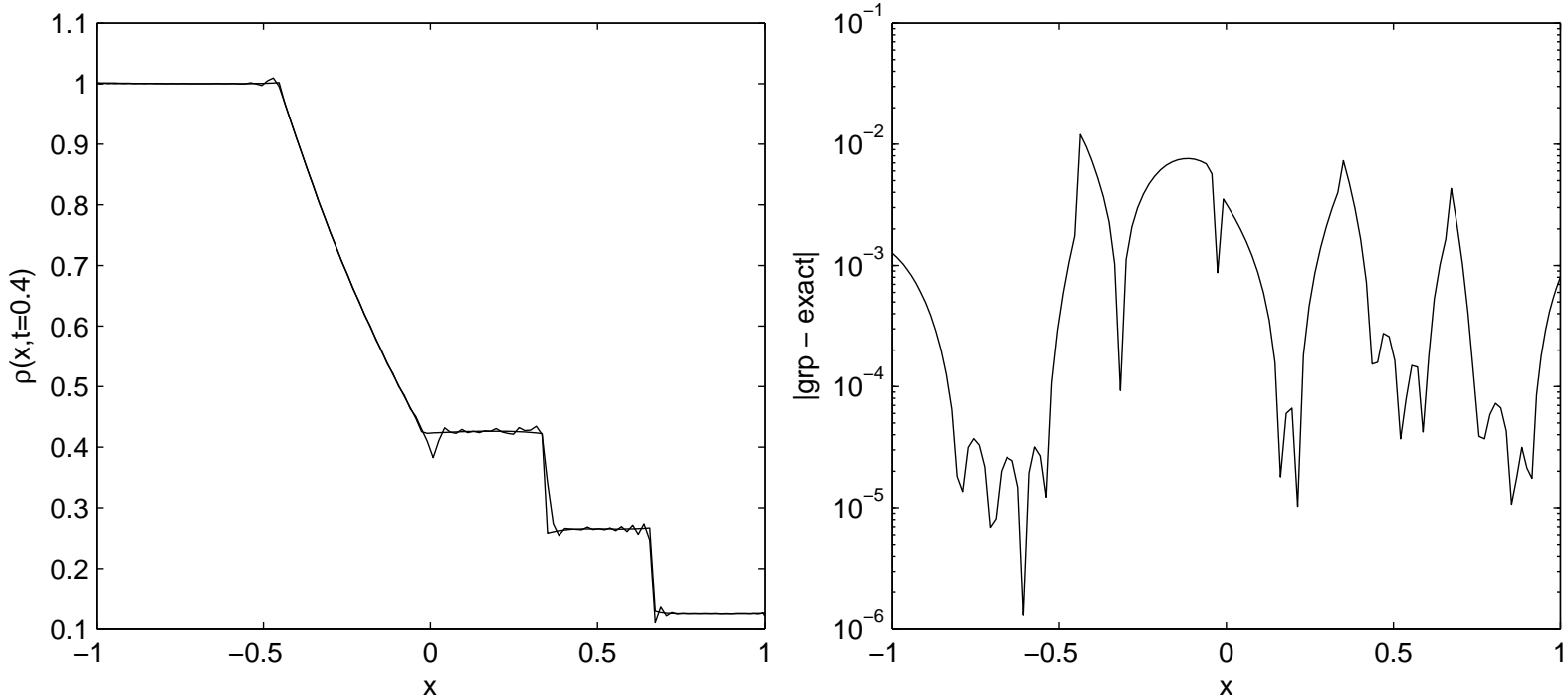
Fourier Gegenbauer Reprojection.

GRP - inaccurate edge detection



Edges located at $x = -0.48$ and $x = 0.5$

Gegenbauer Reprojection Example

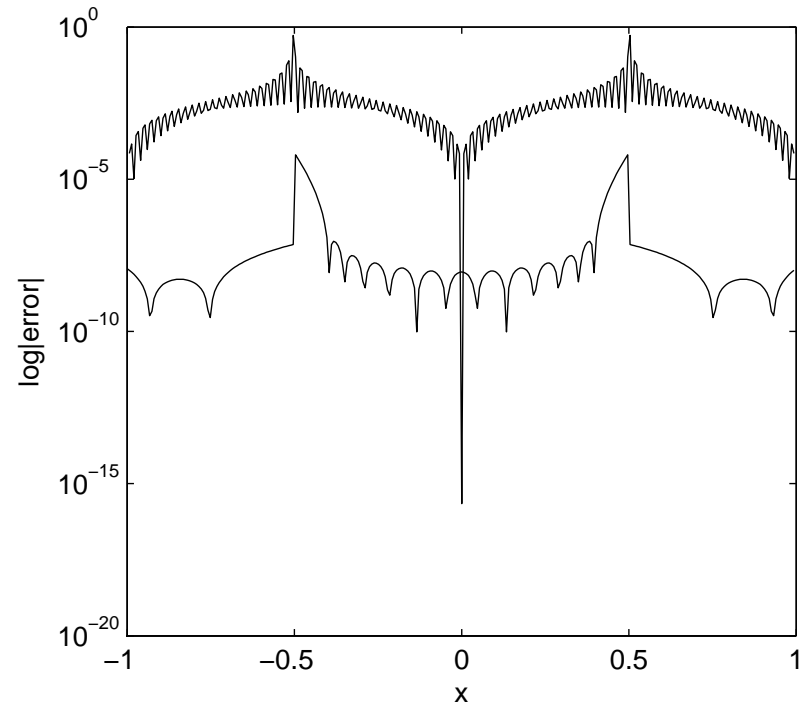
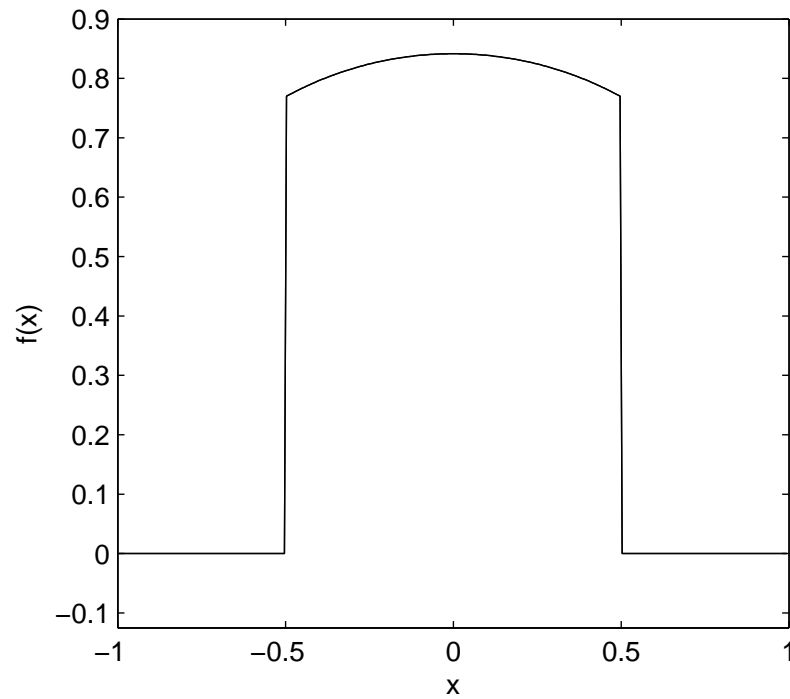


Chebyshev spectral viscosity and GRP postprocessed Euler density solution at time $t = 0.4$.

Freud Reprojection

- Ψ_ℓ^λ , Freud polynomials
- $w(x) = e^{-cx^{2\lambda}}$
- $c = \ln \epsilon_M$ where ϵ_M is machine epsilon
- not known in closed form - calculated via a three term recurrence relationship; weights not known - quadrature or Stieltjes procedure
- accurate edge detection critical
- $\lambda = \text{round} \left(\sqrt{N(b-a)/2} - 2\sqrt{(2)} \right)$
- $m_i = N(b-a)/8$ (with earlier truncation possible)

Freud Reprojection Example

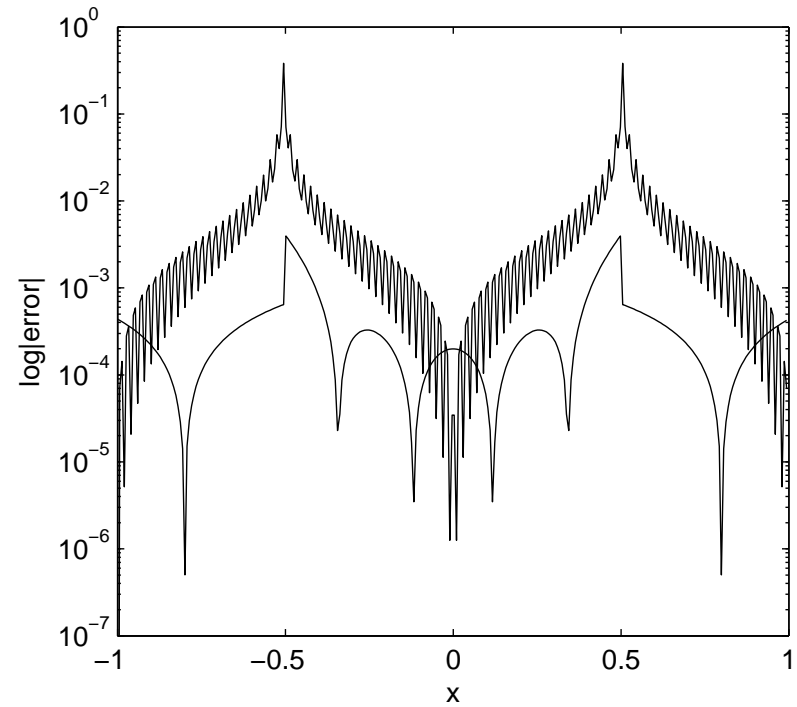
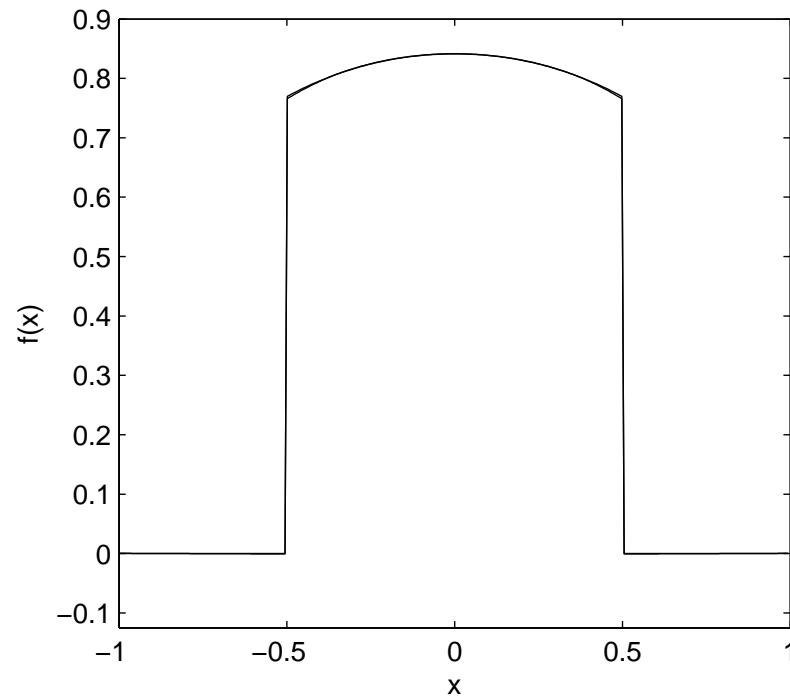


Fourier Freud Reprojection.

Inverse Polynomial Reprojection

- reprojection methods are referred to as direct methods as they compute the reprojection coefficients directly from the spectral expansion coefficients a_k (or function values)
- inverse methods compute the reprojection coefficients by solving a (ill-conditioned) linear system of equations, $Wg = a$
- yield a unique reconstruction using any polynomial reconstruction basis functions
- needs the exact value of the function being approximated at equally spaced grid points in order to calculate the spectral expansion coefficients $a \Rightarrow$ useless in PDE problems
- accurate edge detection critical

Inverse Polynomial Reprojection Example



IPRM postprocessed Fourier approximation.

DTV Filtering

- Originated in Image Processing
- Formulates a minimization problem which leads to a nonlinear Euler-Lagrange PDE to be solved
- Works with point values in physical space and not with the spectral expansion coefficients
- Built in edge detection
- computationally efficient
- no claim of restoring spectral accuracy

DTV Filtering

- Neighborhood of a node α : $N_\alpha = \{\beta \in \Omega \mid \beta \sim \alpha\}$
- Graph variational problem - minimize the fitted TV energy

$$E_\lambda^{TV}(u) = \sum_{\alpha \in \Omega} |\nabla_\alpha u|_a + \frac{\lambda}{2} \sum_{\alpha \in \Omega} (u_\alpha - u_\alpha^0)^2$$

- unique solution - the solution of the nonlinear restoration equation

$$\sum_{\beta \sim \alpha} (u_\alpha - u_\beta) \left(\frac{1}{|\nabla_\alpha u|_a} + \frac{1}{|\nabla_\beta u|_a} \right) + \lambda(u_\alpha - u_\alpha^0) = 0$$

- u^0 - spectral approximation containing the Gibbs oscillations
- λ the user specified fitting parameter
- regularized location variation (strength function) at node α

$$|\nabla_\alpha u|_a = \left[\sum_{\beta \in N_\alpha} (u_\beta - u_\alpha)^2 + a^2 \right]^{1/2}.$$

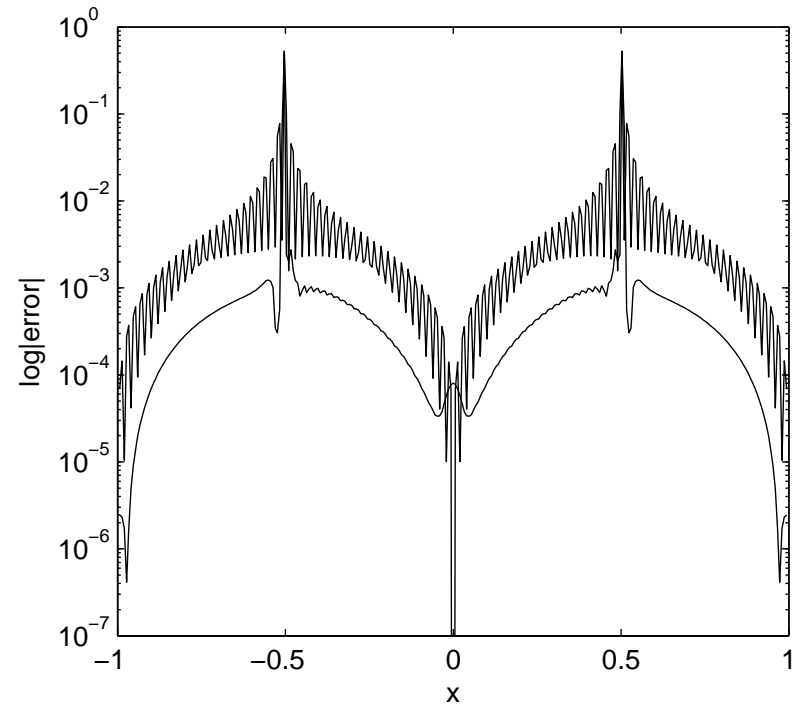
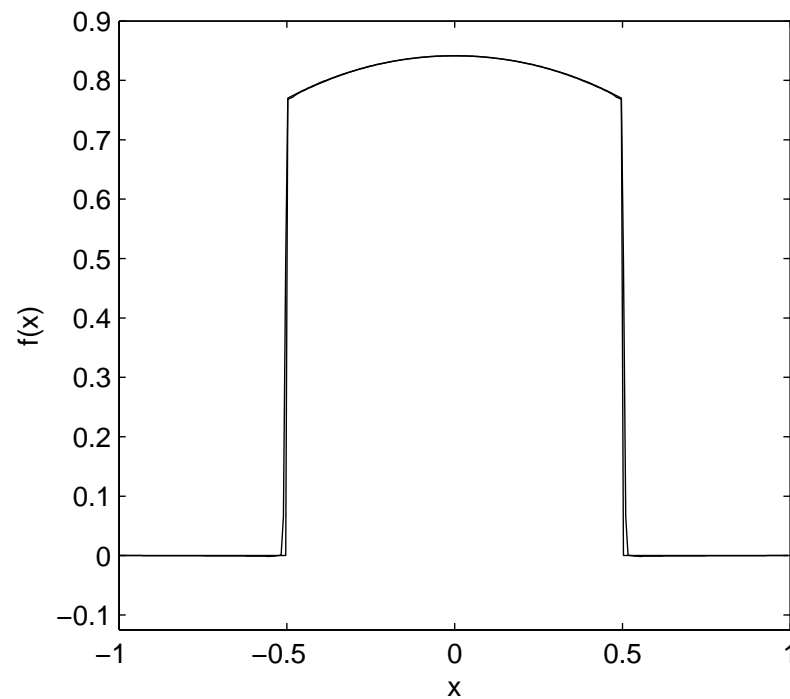
Solving the Restoration Equation

- linearized Jacobi iteration
- time marching with a preconditioned restoration equation

$$\frac{du_\alpha}{dt} = \sum_{\beta \sim \alpha} (u_\alpha - u_\beta) \left(1 + \frac{|\nabla_\alpha u|_a}{|\nabla_\beta u|_a} \right) + \lambda |\nabla_\alpha u|_a (u_\alpha - u_\alpha^0).$$

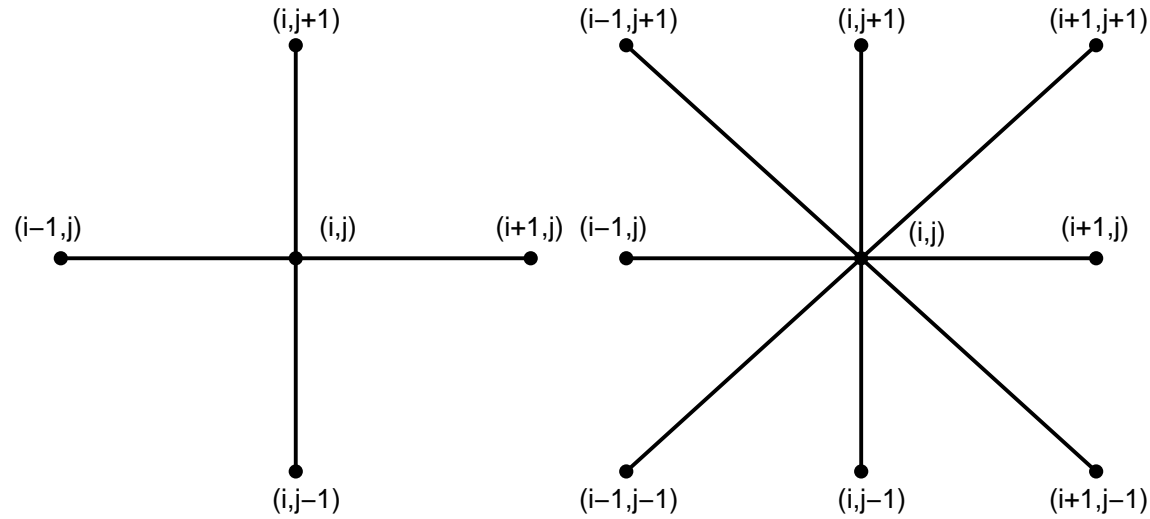
- about 100 time steps are required to approach a steady state
- stronger oscillations - small fitting parameter (< 10)
- weaker oscillations - larger value of the fitting parameter

DTV Filtering example



DTV postprocessed Fourier Approximation.

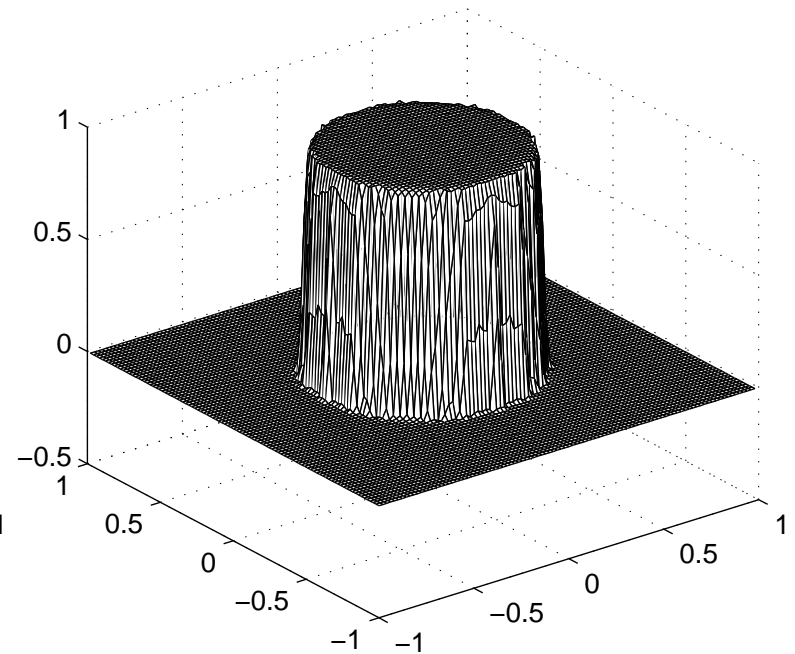
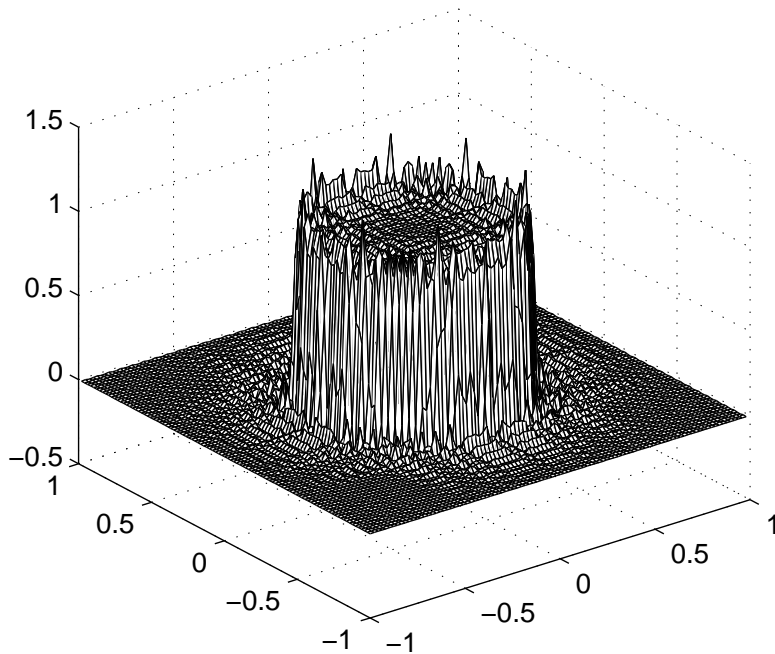
DTV Filtering - 2d



In 2d there is more than one way to define N_α :

1. $N_\alpha^4 = \{\alpha_{i,j+1}, \alpha_{i+1,j}, \alpha_{i,j-1}, \alpha_{i-1,j}\}$
2. $N_\alpha^8 = \{\alpha_{i,j+1}, \alpha_{i+1,j+1}, \alpha_{i+1,j}, \alpha_{i+1,j-1}, \alpha_{i,j-1}, \alpha_{i-1,j-1}, \alpha_{i-1,j}, \alpha_{i-1,j+1}\}$.

DTV Filtering - 2d example



DTV postprocessed Fourier Approximation of

$$f(x) = \begin{cases} 1 & x^2 + y^2 \leq \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

Edge detection free postprocessing

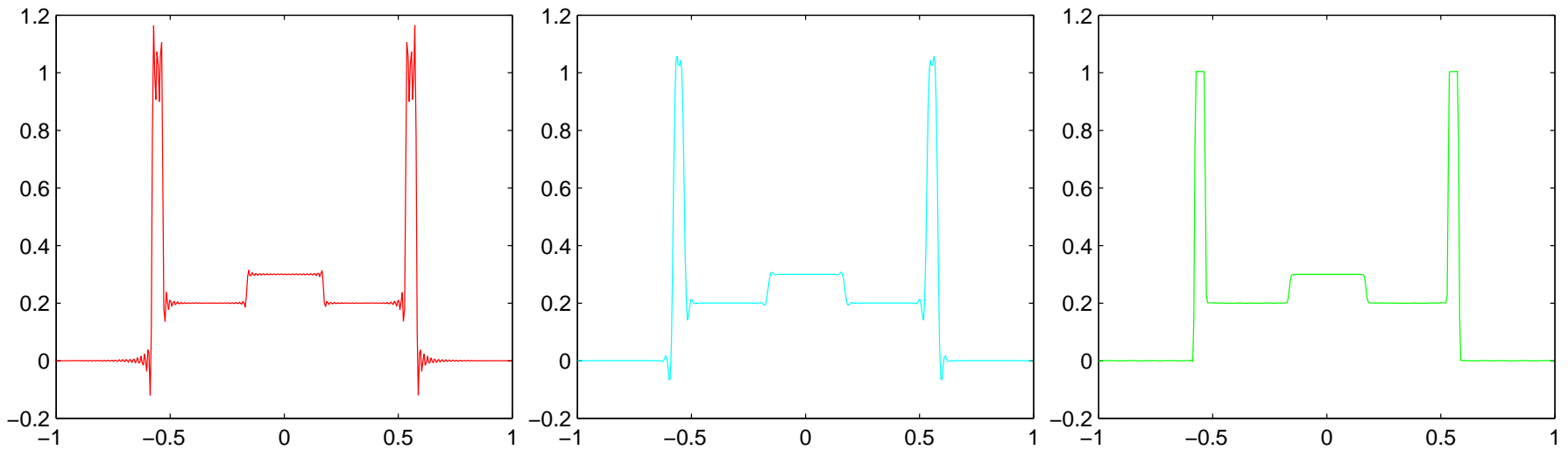
- DTV filtering near discontinuities
- Spectral filtering away from discontinuities
- normalized DTV strength function

$$\mathcal{S} = \frac{|\nabla_{\alpha} u|_a}{\max |\nabla_{\alpha} u|_a}$$

determine which is applied

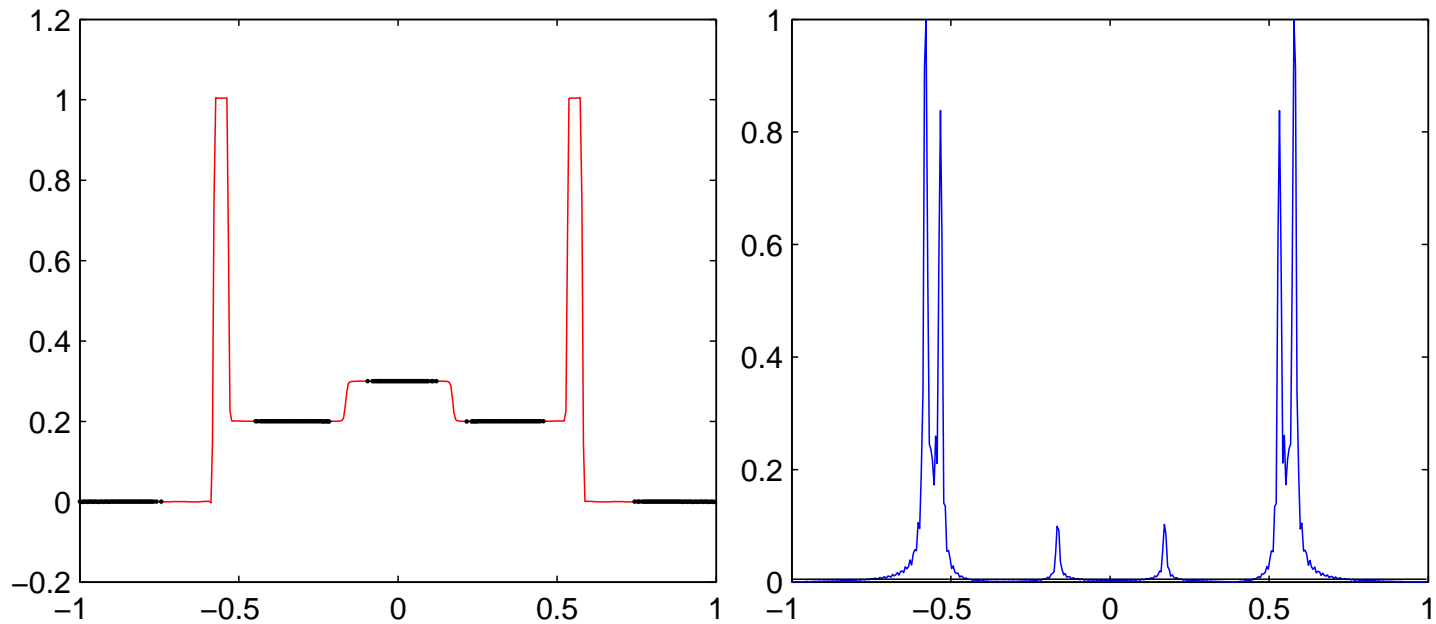
- If $\mathcal{S}[f(x)] < \mathcal{S}_{\max}$ the spectral filter is applied at x .

Hybrid, Shepp-Logan phantom slice



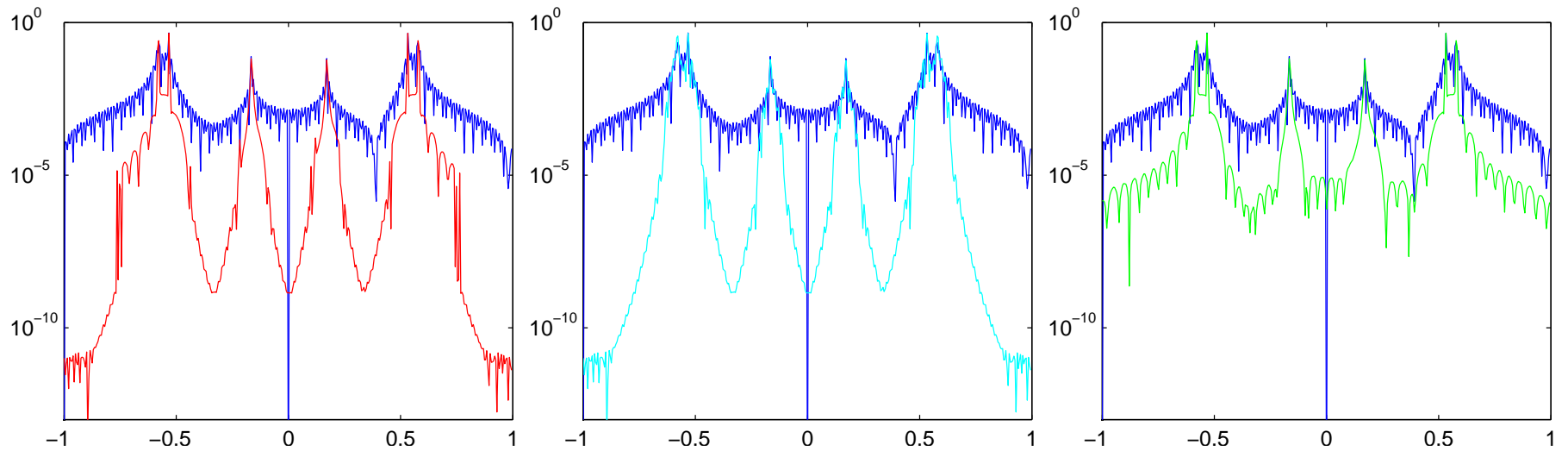
Left: Fourier approximation. Middle: weak spectral filter.
Right: DTV filtered.

Hybrid, Shepp-Logan phantom slice



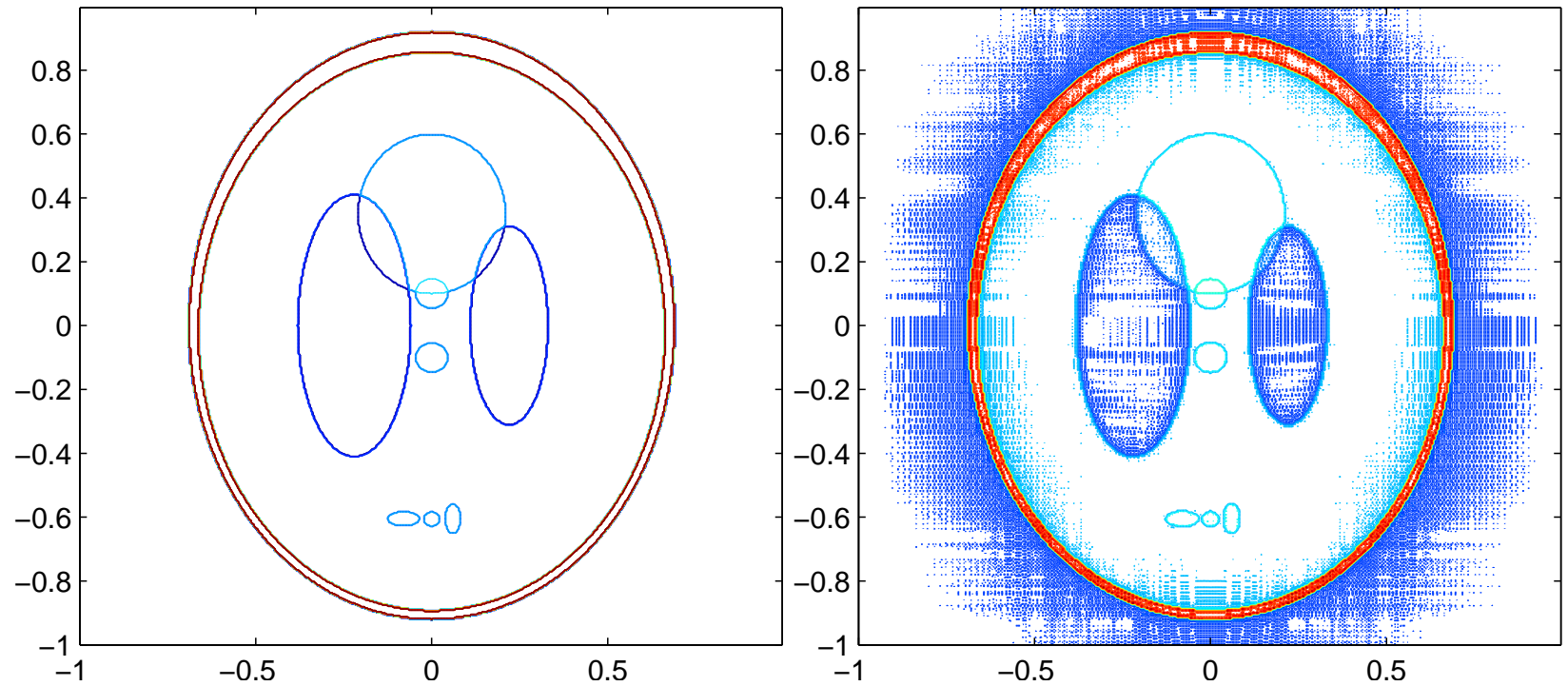
Left: hybrid postprocessed. Right: DTV normalized strength function.

Hybrid, Shepp-Logan phantom slice



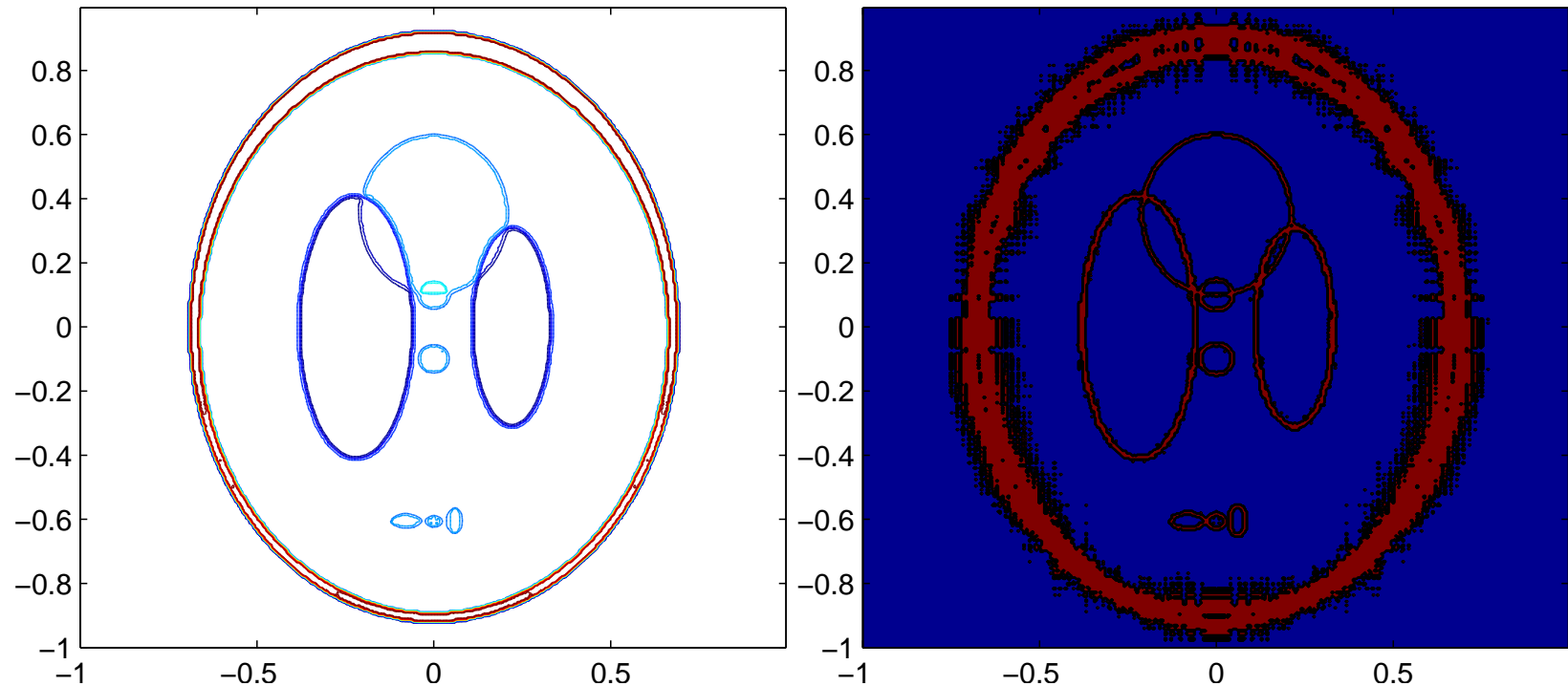
Left: Hybrid error. Middle: spectral filter error. Right: DTV filter error.

Hybrid, Shepp-Logan phantom



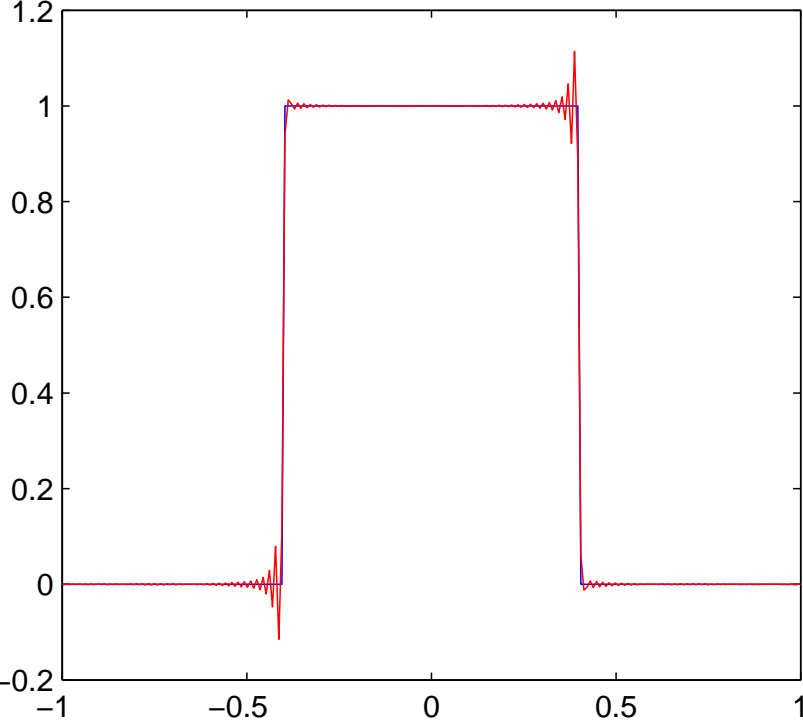
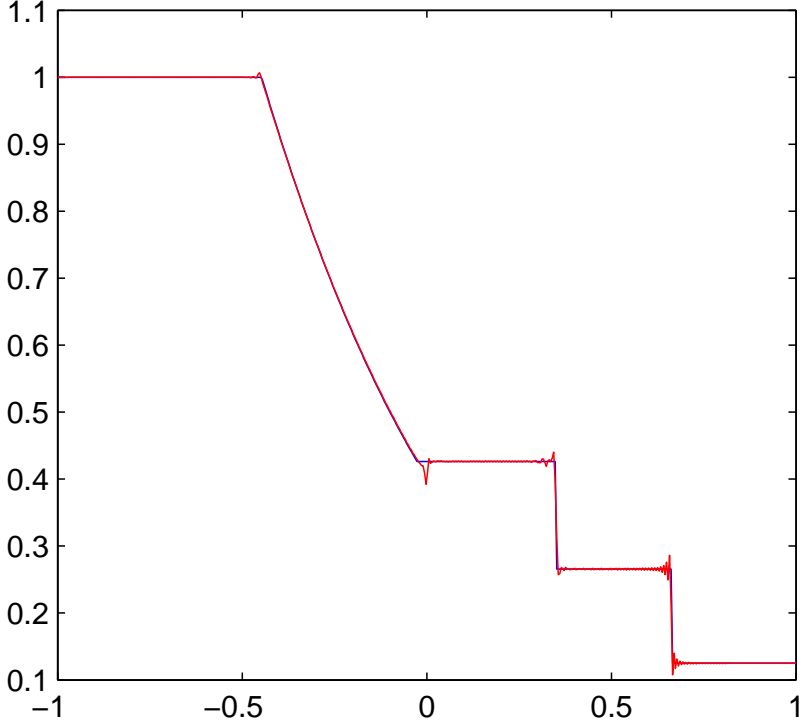
Left: exact phantom. Right: Fourier approximation.

Hybrid, Shepp-Logan phantom

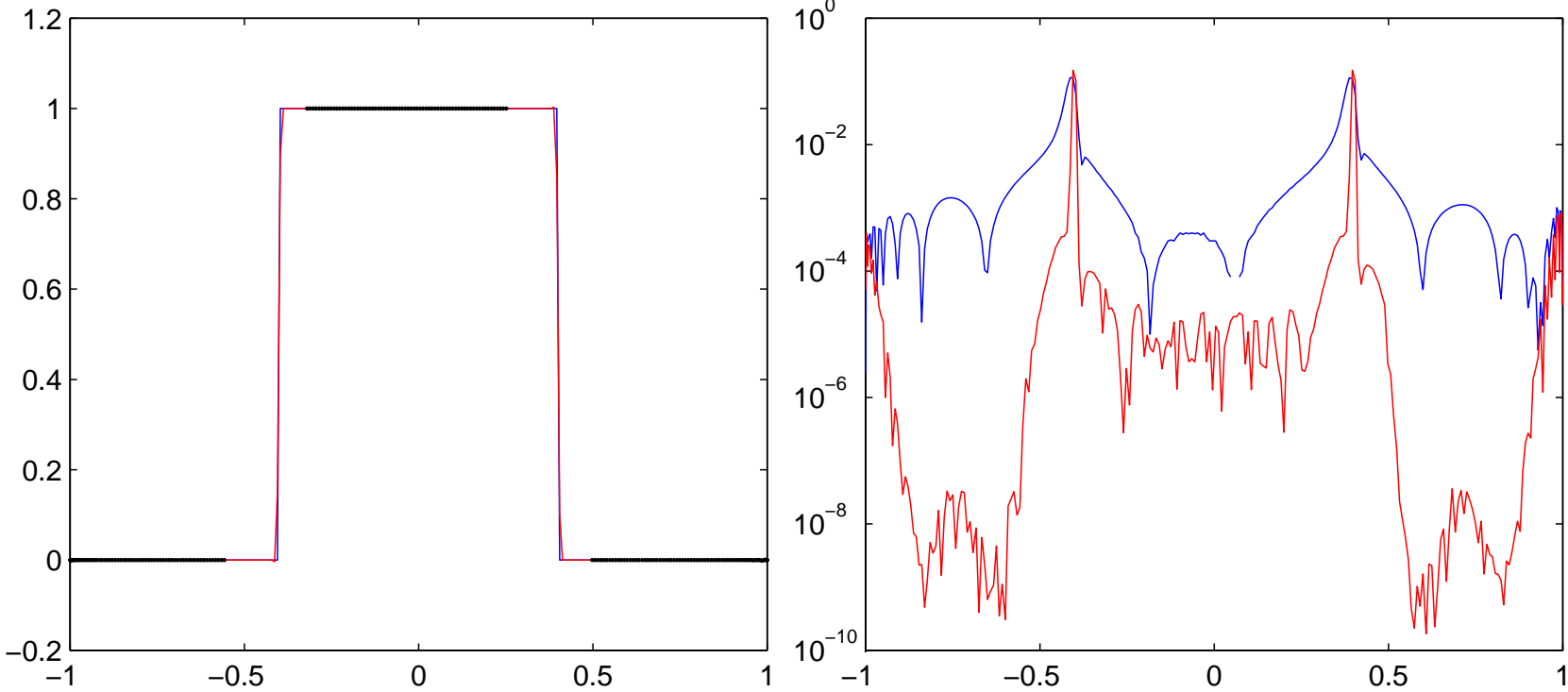


Left: hybrid postprocessed. Right: DTV application area (red), spectral filter (blue).

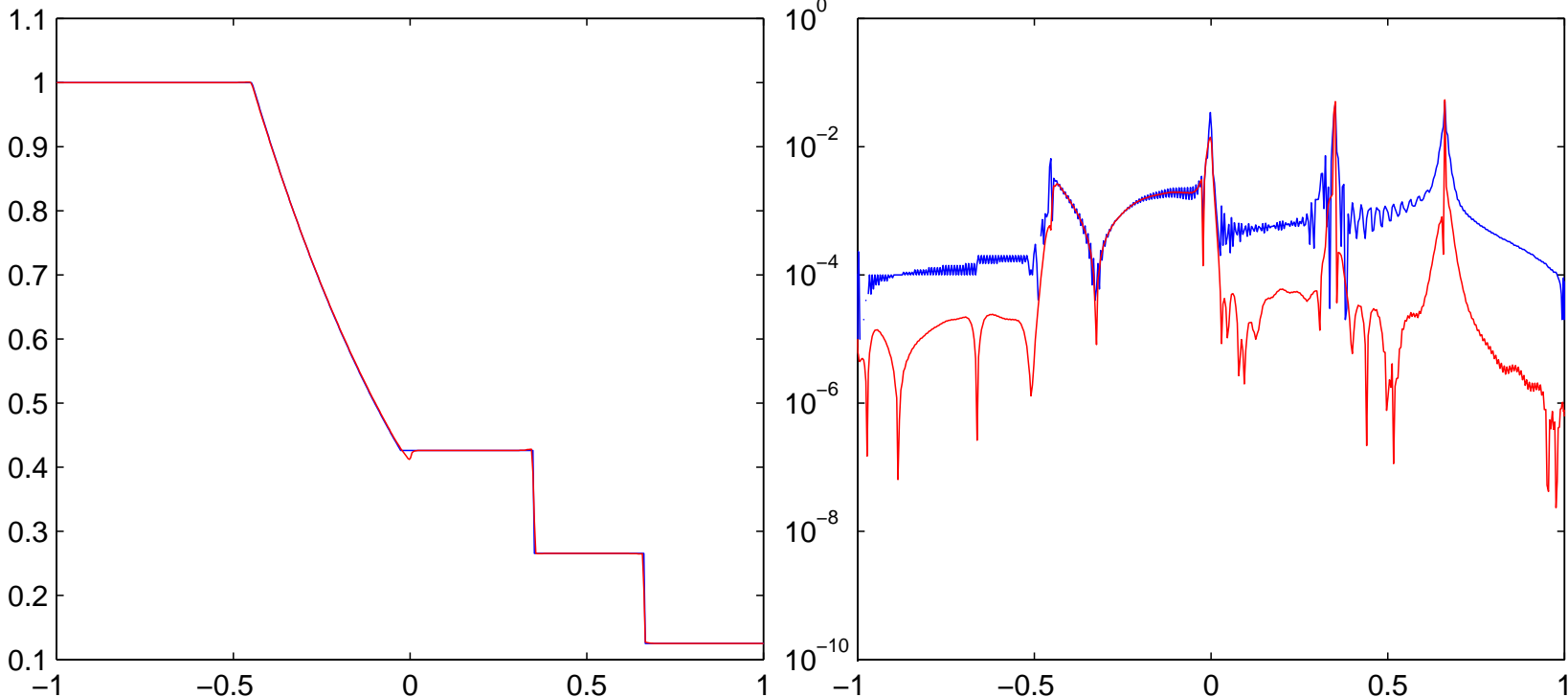
Chebyshev Pseudospectral



Chebyshev Pseudospectral

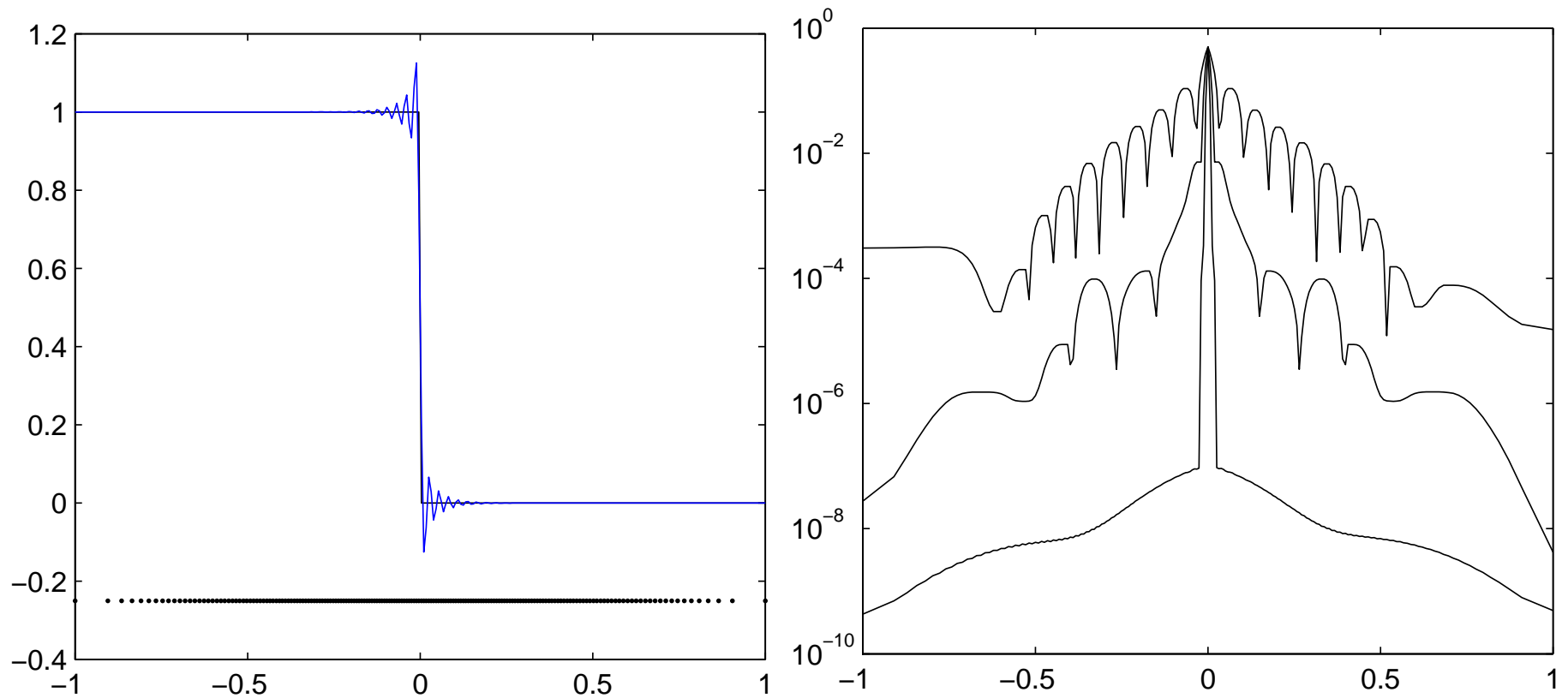


Chebyshev Spectral Viscosity



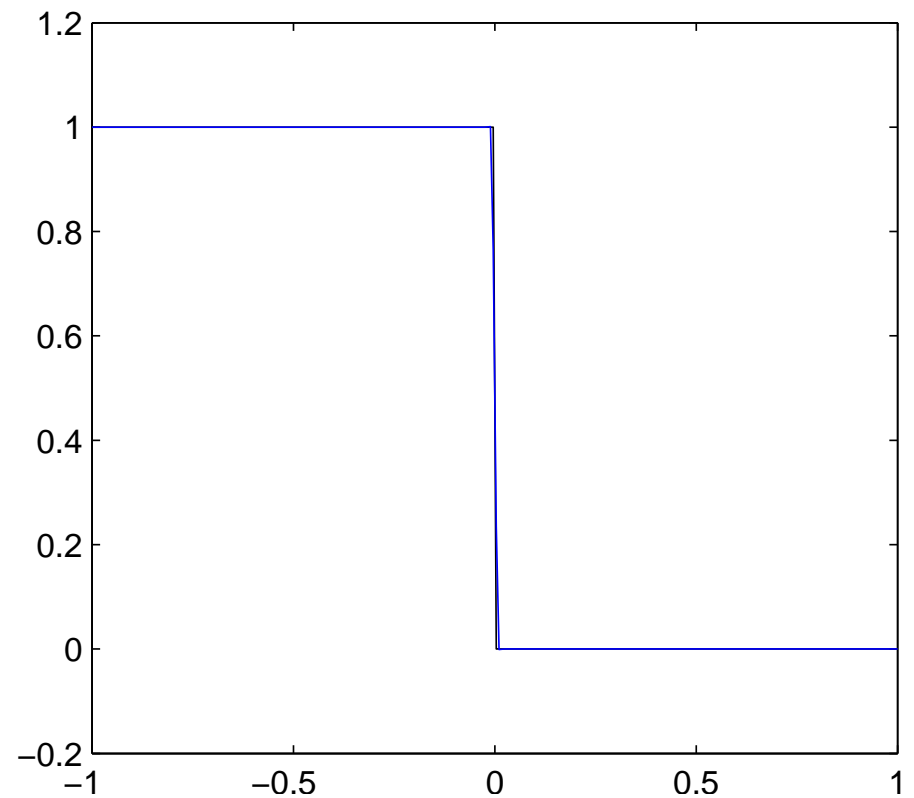
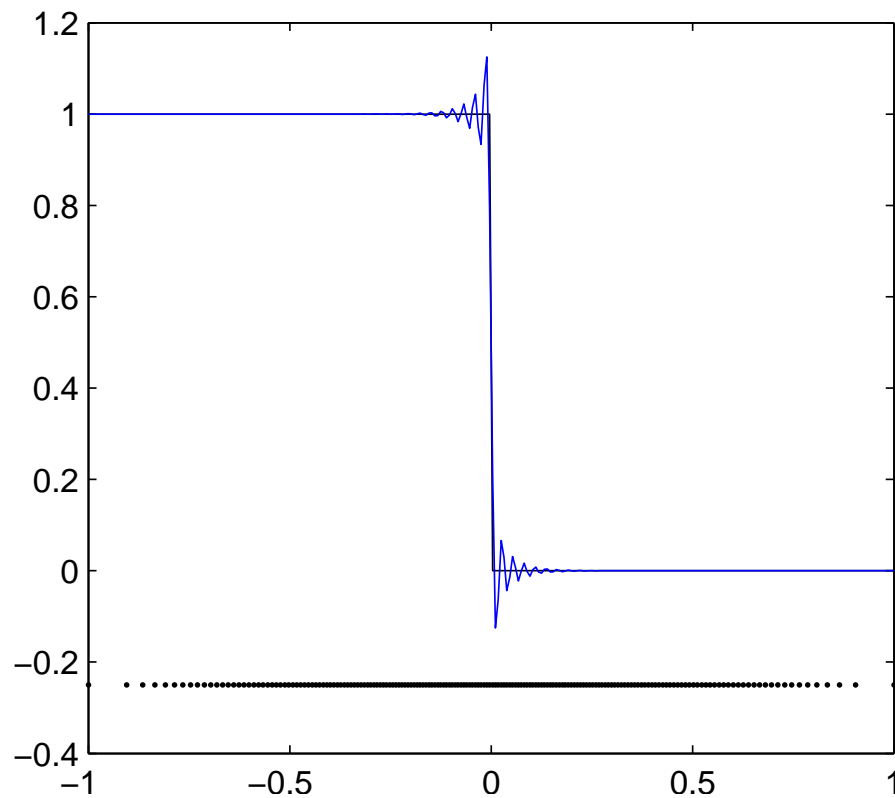
DTV Filter - RBF

RBF DTV example



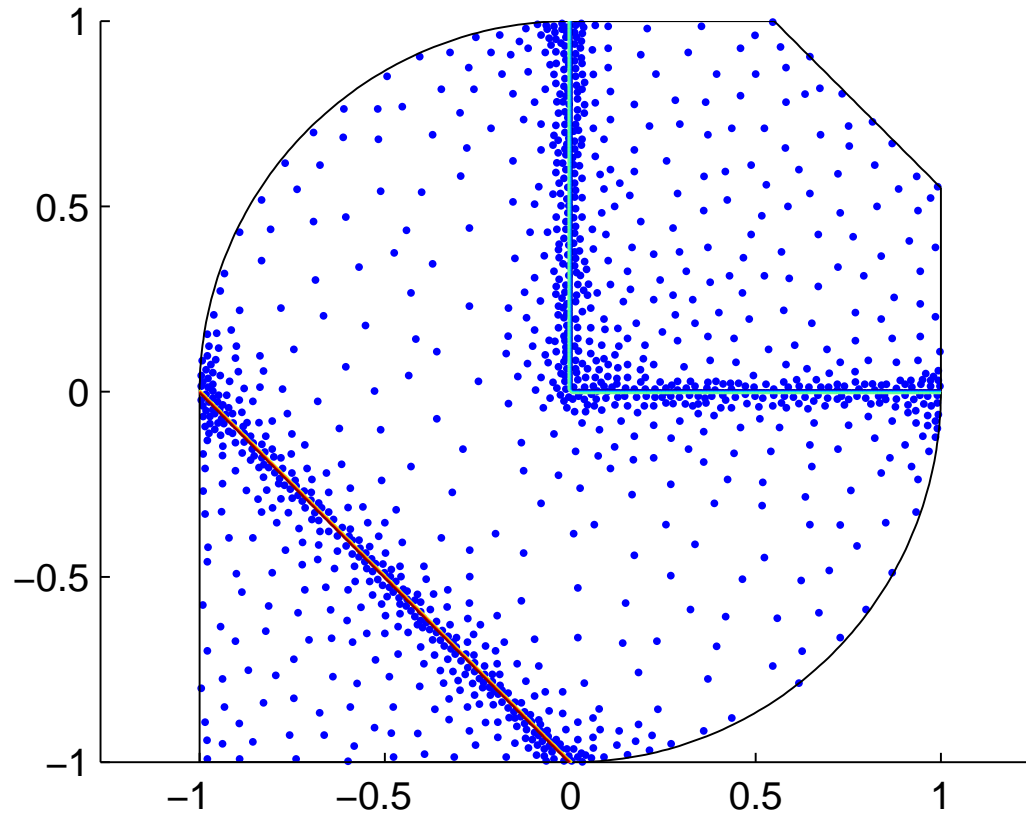
Left: $N = 79$ MQ RBF, Right: DTV postprocessed
convergence $N = 39, 79, 159$.

RBF DTV example



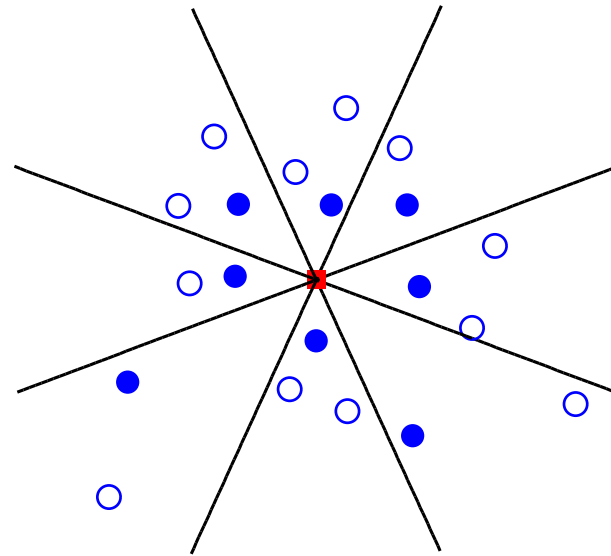
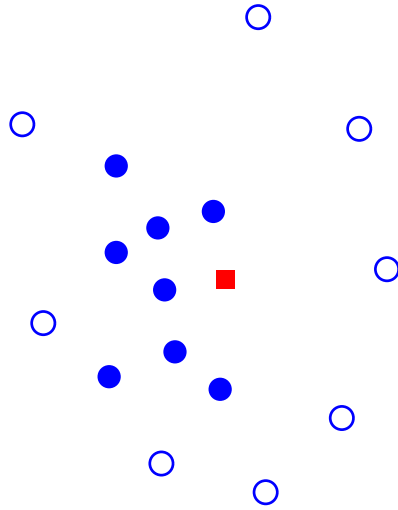
Left: $N = 79$ MQ RBF, Right: DTV postprocessed $N = 79$.

2d Complex Domain - Scattered Data

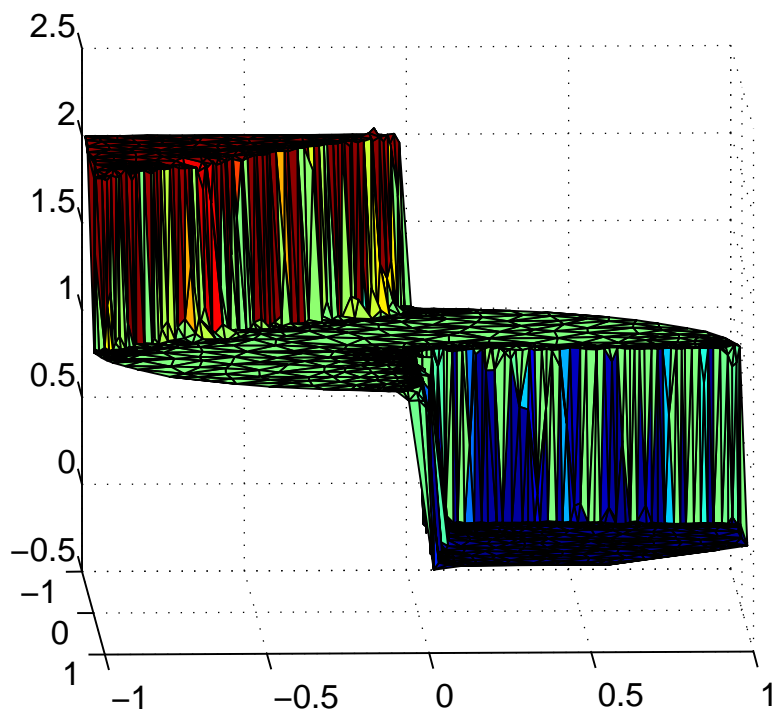
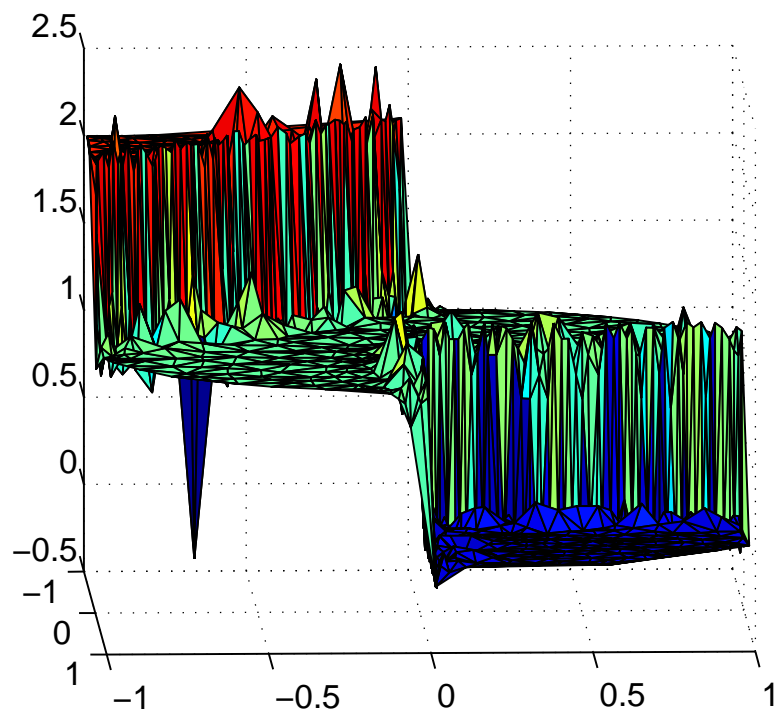


$N = 1126$ scattered points on a complexly shaped domain.

2d Neighborhoods N_α



2d RBF DTV example



Summary

method	edge detection	parameters	spectral accuracy	large $\kappa(A)$	noise
spectral filter		ρ	\sim		✓
DTV		λ			✓
Padé	\sim	M, N_c		✓	✓
GRP	✓	λ, m_i	✓		✓
FRP	✓		✓		✓
IPRM	✓	m_i	✓	✓	