A Numerical Study of Generalized Multiquadric Interpolation

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Radial Basis Functions

- First studied by Roland Hardy - 1968

$$s(x) = \sum_{j=1}^{N} \lambda_j \phi(\| x - x_j^c \|_2, \epsilon)$$

- Allow for scattered data sites to be easily worked with
  - Topographical surfaces
  - Intricate three-dimensional shapes

The most popular RBF that is used in applications today is the Multiquadric (MQ)

$$\phi(r) = \sqrt{1+\varepsilon^2 r^2} = (1+\varepsilon^2 r^2)^{1/2}$$

- Properties of the MQ are well-known [2]

A related RBF with properties not as well-known is the Generalized Multiquadric (GMQ)

$$\phi(r) = (1+\varepsilon^2 r^2)^{\beta} \qquad \beta = \cdots \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2} \cdots$$

- Researchers have recently suggested, but not proven, that the GMQ has desirable properties for β with non-half-integer powers.

GMQ - Suggested Values for β

- Wang and Liu [3]: β = 1.03
- Xaio and McCarthy [4]: β = 1.99
- Kansa [1]: β = 5/2

The behavior of β and the shape parameter, ϵ, are still unknown at the conclusion of their research.

Condition Number vs. Accuracy

- Condition number - a measure of how well a problem will be accurately approximated by a numerical algorithm

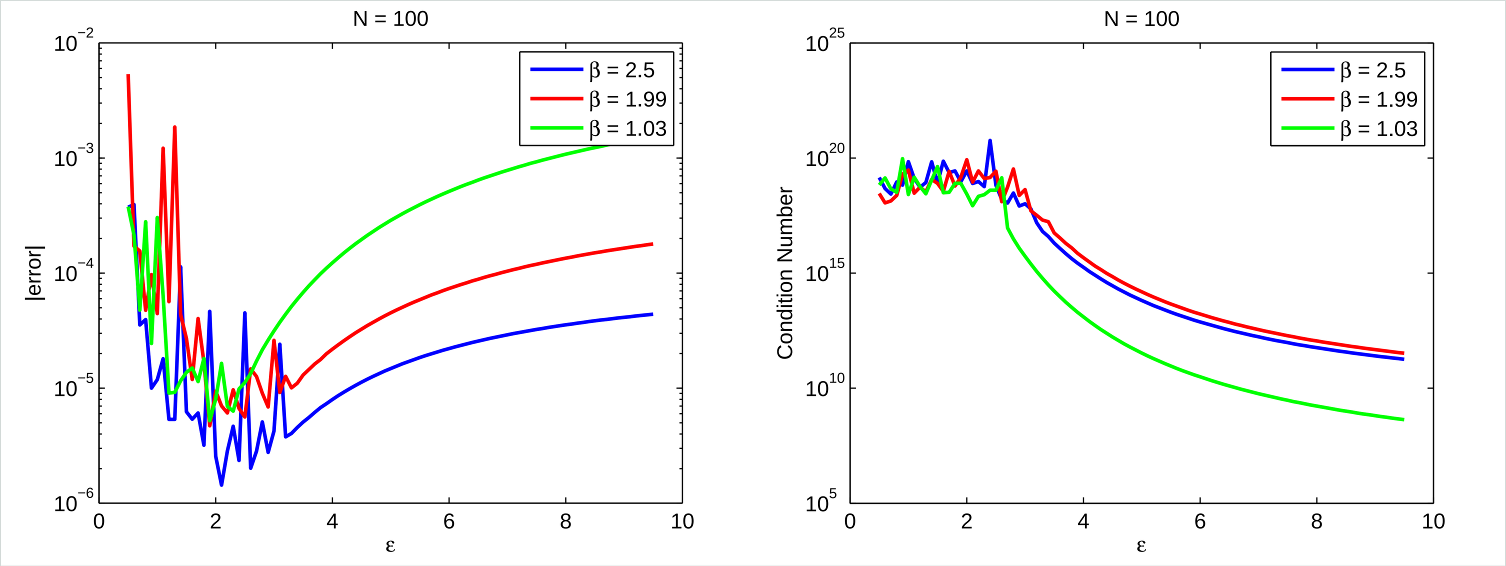
$$f(x) = e^{x^3} + cos(2x)$$

- When β increases, the optimal shape parameters increase
- Minimum error is approximately 10^{-5} for all values of β
  - Condition numbers increase as shape parameters decrease
  - Shape parameter must be large for GMQ to be well-conditioned
  - Small shape parameters are required to obtain good accuracy

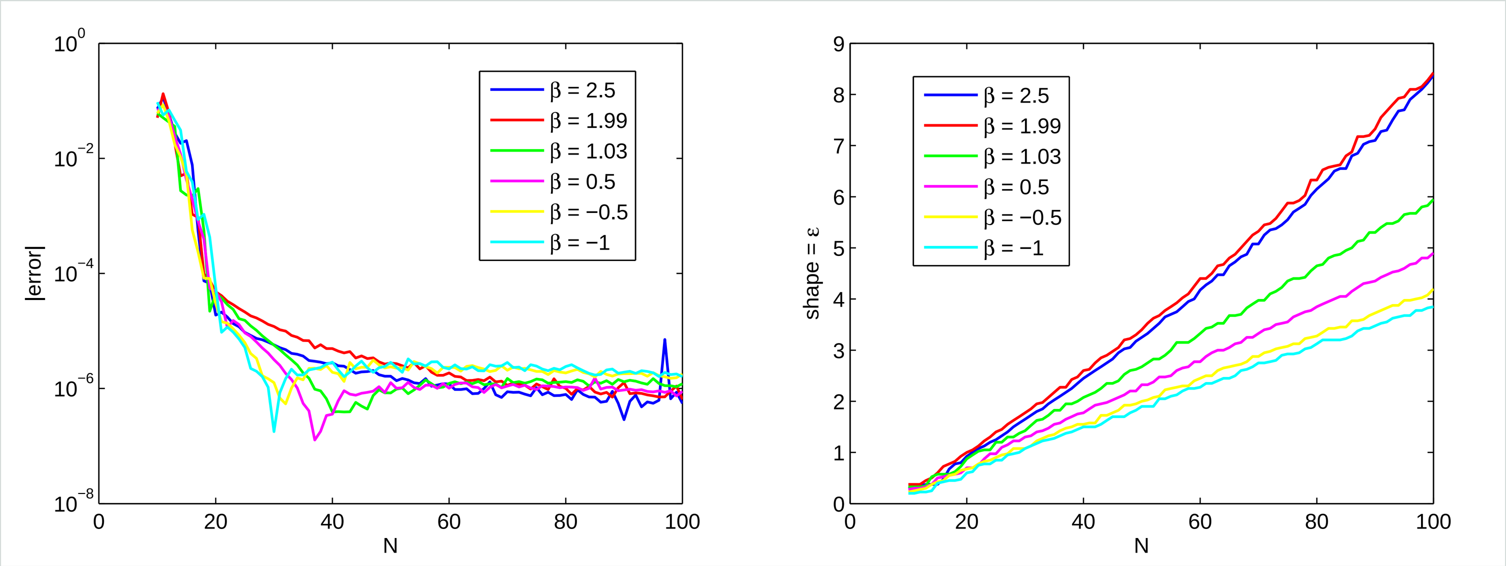
Both instances can obviously not occur at the same time.

- Uncertainty Principle - the more favorably valued one quantity is the less favorably valued the other is
  - Impossible to have good accuracy and conditioning at the same time.

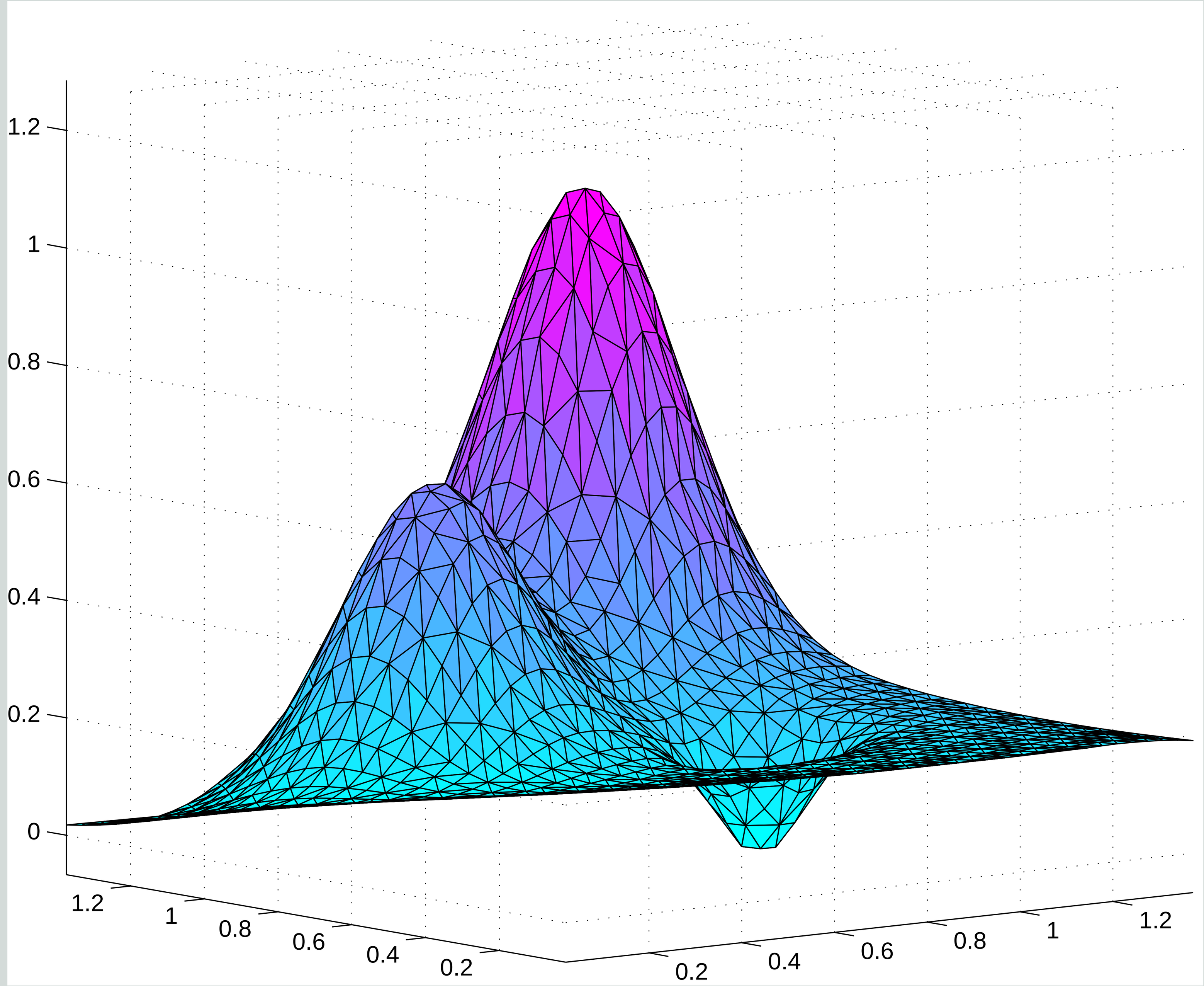
Condition Number versus Accuracy



Searching for Optimal β



Franke Function Generated Using GMQ Interpolation



Searching for Optimal β

Minimum number of centers, N, over a range of shape parameters needed to achieve a targeted accuracy of 10^{-5}:

TABLE 1: N and Shape

β	Smallest N	ϵ
2.5	25	1.25
1.99	32	1.94
1.03	27	1.29
0.5	25	0.98
-0.5	23	0.82
-1	21	0.63

Franke Function

Topographical map created in Matlab with the GMQ.

- Scattered data approximation methods
  - N = 618 centers
  - M = 930 evaluation points

TABLE 2: Franke Function

β	ϵ	$\kappa(\beta)$	Max Error
2.5	3.5	1.6409e+018	5.534104481563773e-006
1.99	4	2.4323e+018	1.790357816333632e-005
1.03	3	1.2784e+018	5.432703227947755e-006
0.5	2	7.2530e+018	8.662307697648863e-005
-0.5	2.25	6.3693e+018	4.832552523308109e-006
-1	2	1.2210e+018	7.464702949815105e-006

Summary

- The MQ does not produce a more accurate approximation than the GMQ.
  - Optimal values for β are possibly problem dependent.
- It is important to choose a shape parameter that allows for the GMQ interpolation to be critically conditioned

References

[1] E. J. Kansa. Numerical simulation of two-dimensional combustion using mesh-free methods. *Engineering Analysis with Boundary Elements*, 2009.

[2] S. Sarra. Radial basis function interpolation. *Submitted to SIURO*, 2009.

[3] JG Wang and GR Liu. On the optimal shape parameters of radial basis functions used for 2-d meshless methods. *Comput. Meth. Appl. Mech. Eng.*, 191:2611–2630, 2002.

[4] J. R. Xaio and M. A. McCarthy. A local heaviside weighted meshless method for two-dimensional solids using radial basis functions. *Computational Mechanics*, 31:301–315, 2003.