Chebyshev Interpolation: an interactive tour

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1 Introduction

Most areas of numerical analysis, as well as many other areas of Mathematics as a whole, make use of the Chebyshev polynomials. In several areas, e.g. polynomial approximation, numerical integration, and pseudospectral methods for partial differential equations, the Chebyshev polynomials take take a significant role. In fact, the following quote has been attributed to a number of distinguished mathematicians:

> "The Chebyshev polynomials are everywhere dense in numerical analysis."

In this manuscript we make use of Java applets to interactively explore some of the classical results on approximation using Chebyshev polynomials. We also discuss an active research area that uses the Chebyshev polynomials. References [14, 15] are devoted to the Chebyshev polynomials may be consulted for more detailed information than we provide in this brief presentation.

2 Overview of the Applets

The applets used are the:

Chebyshev polynomial (CP) applet. The CP applet plots the degree zero to degree twelve Chebyshev polynomial. The degree of the polynomial can be changed by changing the value of the slider at the bottom of the applet.

Chebyshev approximation (CA) applet. The applet visualizes the interpolatory, the continuous, the discrete, and the filtered Chebyshev

approximations of several functions. The functions are described in section [8] below. The functions can be selected in the applet from the **Functions menu** at the top of the applet. The applet contains two windows. In the left window the exact function is plotted in black. Up to two approximations can be plotted (the first in blue and the second in red) and compared by making selections from the **Approximations menu**. By default the approximation error is displayed in the right window. The magnitude of the Chebyshev coefficients can be displayed in the right window by selecting *plot coefficients* from the **Options menu**. The y-axis of the error or coefficient plot can be displayed on a log scale by selecting *semiLogY* from the Options menu. The *plot CGL points* option marks the exact value of the interpolated function in the left window with a green dot at the interpolation sites. The *connect (right)* option on the options menu, when left unchecked, allows the error plot to be displayed without the points on the graph being connected. This sometimes produces a more pleasing plot on a semilog scale when values are near zero. A parameters dialog can be displayed by pressing the *parameter dialog* option from the options menu. From the dialog the order of the exponential filter may be changed as well as the number of evenly spaced points NP at which the interpolants are evaluated.

Runge phenomenon (RP) applet. The RP applet illustrates the divergence of equidistant polynomial interpolation by using a classic example.

Exponential filter (EF) applet. The EF applet plots the value of the exponential filter of degrees 2 to 32 by changing the value of the slider at the bottom of the applet. The value of the filter is plotted versus the normalized coefficient number n/N. The applet visualizes how much each Chebyshev coefficient is modified by the filter.

3   Chebyshev Polynomials

The *Chebyshev Polynomials* (of the first kind) are defined as

$$T_n(x) = \cos\left[n\arccos(x)\right]. \tag{1}$$

They are orthogonal with respect to the weight $w(x) = (1 - x^2)^{-1/2}$ on the interval $[-1, 1]$. Intervals $[a, b]$ other than $[-1, 1]$ are easily handled by the change of variable $x \to \frac{1}{2}\left[(b-a)x + a + b\right]$. Although not immediately

evident from definition (1), each $T_n(x)$ is a polynomial of degree n. From definition (1) we have that $T_0(x) = \cos(0) = 1$ and $T_1(x) = \cos(\arccos x) = x$. For $n \geq 1$ we have the *triple recursion relation*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{2}$$

which can be shown to be true using basic trig identities. Using equation (2) we see

$$
\begin{aligned}
T_2(x) &= 2x(x) - 1 = 2x^2 - 1 \\
T_3(x) &= 2x(2x^2 - 1) - x = 4x^3 - 3x \\
T_4(x) &= 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1 \\
&\vdots
\end{aligned}
$$

and that the Chebyshev polynomials are indeed polynomials of degree n.

Applet Activity. What do the Chebyshev polynomials look like? The Chebyshev polynomials of degree $k = 0, 1, \ldots, 12$ can be plotted in the CP applet below. Notice that $|T_n(x)| \leq 1$. Since $T_n(x)$ is a degree n polynomial we can observe as expected that it has n zeros, which in this case are real and distinct and located in $[-1, 1]$.

Exercise.[Zeros of the Chebyshev polynomials.] Show that the zeros of $T_n(x)$ are

$$x_k = \cos\left(\frac{\pi(2k+1)}{2n+2}\right), \quad k = 0, 1, \ldots, n \tag{3}$$

The zeros are known as the ***Chebyshev-Gauss (CG) points***.

4   Continuous Chebyshev Expansion

The *infinite continuous Chebyshev series expansion* is

$$f(x) \approx \sum_{n=0}^{\infty}{}' \alpha_n T_n(x) \tag{4}$$

where
$$\alpha_n = \frac{2}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} \, f(x) \, T_n(x) dx. \tag{5}$$

Truncating the series after $N + 1$ terms, we get the *truncated continuous Chebyshev expansion*
$$\mathcal{S}_N(x) = \sum_{n=0}^{N} {}' \alpha_n T_n(x). \tag{6}$$

The single prime notation in the summation indicates that the first term is halved. There are several functions in which the integral for the coefficients α_n can be evaluated explicitly, but this is not possible in general. Examples included in the CA applet for which a continuous truncated expansion can be derived are the sign function (32), $f_4(x) = \sqrt{1 - x^2}$, and $f_5(x) = |x|$.

The conditions which must be placed on f to ensure the convergence of the series (4) depend on the type of convergence to be established: pointwise, uniform, or L^2. At the lowest level, the series (4) converges pointwise to f at points where f is continuous in $[-1, 1]$ and converges to the left and right limiting values of f at any of a finite number of jump discontinuities in the interior of the interval.

Applet Activity. The sign function in the CA applet has a jump discontinuity at $x_0 = 0$ and has the limiting values on each side of the discontinuity of $f(x_0+) = 1$ and $f(x_0-) = -1$. Thus the series converges to zero at this point, i.e.
$$\mathcal{S}_N(x_0) \approx \frac{1}{2}[f(x_0^+) + f(x_0^-)]$$

for sufficiently large N. In the applet select the sign function from the Functions menu and check the blue *continuous, S* option on the Approximation menu. Using the slider at the bottom of the applet, slowly adjust N from $N = 7$ to $N = 128$ and observe that the value of $S_N(0)$ is approximately zero.

If $f(x)$ is an even function then $\alpha_k = 0$ for $k = 1, 3, 5, \ldots$. If $f(x)$ is an odd function then $\alpha_k = 0$ for $k = 0, 2, 4, \ldots$.

Applet Activity. The above fact can be observed in the truncated continuous expansion of $f(x) = \sqrt{1 - x^2}$ and $f(x) = |x|$ (even) and $f(x) = sign(x)$ (odd) in the CA applet. For example, select the even function

4

$f(x) = \sqrt{1 - x^2}$ which is labelled as *sqrt* on the Functions menu and select the blue *continuous, S* option on the Approximation menu. Then on the options menu check *plot coefficients* and using the slider slowly adjust N from $N = 7$ to $N = 21$. In the right window observe that $\alpha_k = 0$ for $k = 1, 3, 5, \ldots$. The magnitude of the coefficients can also be viewed with the y-axis scaled logarithmically (*semiLogY* on the options menu). However, in this case the coefficients which are zero are not plotted as $\log(0)$ is undefined.

5 Discrete Chebyshev Expansions

When the integral in (5) can not be evaluated exactly, we can introduce a discrete grid and use a numerical quadrature (integration) formula. Several possible grids, and related quadrature formulas exist. The **Chebyshev-Gauss-Lobatto (CGL)** points

$$x_k = -\cos\left(\frac{k\pi}{N}\right) \quad k = 0, 1, \ldots, N \tag{7}$$

are a popular choice of quadrature points. The CGL points are where the $n-1$ extrema of $T_n(x)$ occur plus the endpoints of the interval $[-1, 1]$.

Applet Activity. Using the CP applet, observe how the extrema of the Chebyshev polynomials are not evenly distributed and how they cluster around the boundary. In the CA applet, the CGL points may be plotted by checking *plot CGL points* on the options menu. Try this with the sign function starting with $N = 9$ and then with increasing N.

Exercise.[Extrema of the Chebyshev polynomials] Show that $T_n(x) = \pm 1$ at the $n-1$ CGL points.

The corresponding CGL quadrature formula is

$$\int_{-1}^{1} \frac{f(x)dx}{\sqrt{1 - x^2}} \approx \frac{\pi}{N} \sum_{j=0}^{N} {}'' f(x_k). \tag{8}$$

If $f(x)$ is a polynomial of degree less than or equal to $2N - 1$, the CGL quadrature formula is exact. This is remarkable accuracy considering that the values of the integrand are only known at the $N + 1$ CGL points. Using the CGL quadrature formula to evaluate the integral in (5), the discrete Chebyshev coefficients a_n are defined to be

$$\alpha_n \approx a_n = \frac{2}{N} \sum_{k=0}^{N} {}'' f(x_k) T_n(x_k) \tag{9}$$

and the *discrete truncated partial sum* is

$$\mathcal{P}_N(x) = \sum_{n=0}^{N} {}' a_n T_n(x). \tag{10}$$

Using definition (9) takes $\mathcal{O}(N^2)$ floating point operations (flops) to evaluate the discrete Chebyshev coefficients. For large N, a better choice is the fast cosine transform (FCT) [4] which takes $\mathcal{O}(N \log_2 N)$ flops. For example, if $N = 1000$, $N^2 = 1,000,000$ while $N \log_2 N < 10,000$. The extreme efficiency of the FCT is one reason for the popularity of Chebyshev approximations in applications.

5.1 Interpolating Partial Sum

Requiring that the approximation be interpolating, i.e., requiring that it satisfy

$$\mathcal{P}_N(x_i) = f(x_i) \quad i = 0, 1, \ldots, N \tag{11}$$

we get the *interpolating partial sum*

$$\mathcal{I}_N(x) = \sum_{n=0}^{N} {}'' a_n T_n(x). \tag{12}$$

The double prime notation in the summation indicates that the first and last terms are halved. The interpolating partial sum would be equal to the truncated series with the coefficients approximated via CGL quadrature except the last coefficient is halved. This is due to the choice of quadrature points. If Gaussian quadrature, which uses the Chebyshev-Gauss (CG) points, had been used instead of CGL quadrature, the interpolating and discrete truncated partial sum would be identical. The CG points are the zeros of $T_n(x)$ and do not include $x = \pm 1$. Chebyshev pseudospectral methods for solving PDEs usually incorporate the CGL points and not the CG points. The

reason for this is that the discrete grid must include the boundary points so that the boundary conditions of the PDE can be incorporated into the numerical approximation.

Applet Activity. Using the applet we can observe the difference between $\mathcal{S}_N$, $\mathcal{P}_N$, and $\mathcal{I}_N$. For example if we use the *sign* function (select sign from the Functions menu) with $N = 11$ (set N using the slider at the bottom of the applet) and plot the CGL points (check *plot CGL points* on the Options menu) we see that $\mathcal{I}_N$ goes through the interpolation sites while $\mathcal{S}_N$ and $\mathcal{P}_N$ do not (On the Approximations menu, select the blue *interpolation, I* and then the red *discrete, P*. Then select the red *continuous, S* to make the next comparison).

Since (12) is a polynomial of at most degree N that satisfies the interpolation condition (11) at $N + 1$ distinct points, a standard result from numerical analysis tells us that $\mathcal{I}_N$ is the unique interpolating polynomial [5, p. 106]. The interpolating polynomial may be written in several equivalent forms: Lagrange, Newton, and Barycentric. For information on the merits of each form, see [1]. The Lagrange form of the interpolating polynomial is

$$\mathcal{I}_N(x) = \sum_{k=0}^{N} f(x_k) L_k(x)$$

where

$$L_k(x) = \prod_{i=0, i \neq k}^{N} \frac{x - x_i}{x_k - x_i} \tag{13}$$

are *cardinal* polynomials that satisfy

$$L_k(x_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases} \tag{14}$$

The Lagrange form gives an error term of the form

$$e_N(x) = f(x) - \mathcal{I}_N(x) = \frac{f^{(N+1)}(\xi(x))}{(N+1)!} \Psi(x) \tag{15}$$

where

$$\Psi(x) = \prod_{j=0}^{N} (x - x_j). \tag{16}$$

The underlying function $f(x)$ is often unknown and the number ξ is only known in simple examples. Thus, $\Psi(x)$ is the only part of the error term which can be controlled. By using the CG or CGL points as interpolation cites, $\Psi(x)$ is made nearly as small as possible [5, p. 507]. On the other hand, it is well known that polynomial interpolation in equally spaced points can be troublesome. The classic example provided by Runge is the function

$$f(x) = \frac{1}{1 + x^2}, \qquad -5 \leq x \leq 5. \tag{17}$$

For the function (17), equidistant polynomial interpolation diverges for $|x| > 3.63$. By using the CGL points (7), which cluster densely around the endpoints of the interval, as interpolation sites the nonuniform convergence (the *Runge Phenomenon*) associated with equally spaced polynomial interpolation is avoided.

Applet Activity. The RP applet illustrates equidistant and Chebyshev interpolation for the Runge example (17). The applet starts with $N = 15$ and equidistant interpolation. Use the slider to increase N and observe that the oscillations near the boundary become larger and that the approximation is good for $|x| < 3.63$. Select the CGL button at the top of the applet and observe that the oscillations near the boundary disappear.

5.2 Aliasing

Introducing a discrete grid leads to *aliasing*. The discrete coefficients can be expressed in terms of the continuous coefficients as

$$a_n = \alpha_n + \sum_{j=1}^{\infty} \left(\alpha_{n+2jN} + \alpha_{-n+2jN} \right). \tag{18}$$

As an example consider the *sign* function with $N = 9$. The difference between the discrete coefficient a_5 and the continuous coefficient α_5 can be quantified by the aliasing relation (18) as

$$
\begin{aligned}
a_5 - \alpha_5 \;=\;& \alpha_{23} + \alpha_{41} + \alpha_{59} + \ldots \\
+\;& \alpha_{13} + \alpha_{31} + \alpha_{49} + \ldots.
\end{aligned}
$$

This relation is a result of the fact that on the discrete grid, T_5 is identical to $T_{23}, T_{41}, T_{59}, \ldots$ and also to $T_{13}, T_{31}, T_{49}, \ldots$ as is illustrated in figure 1. The image was produced with the following Matlab script:

```
 N = 9;    M = 600;
 x = -cos(pi*(1:9)./N);       % extrema and endpoints of T10
xp = linspace(-1,1,M);

 T23 = cos(23*acos(xp));      %  cyan
 T13 = cos(13*acos(xp));      %  green
  T5 = cos( 5*acos(xp));      %  blue
 T5g = cos( 5*acos(x));       %  T_5(x) (red *)

 XGL10 = zeros(1,length(x));  %  CGL pts (open black circles)

 plot(xp,T5,'b',xp,T13,'g',x,T5g,'r*',x,XGL10,'k-o',xp,T23,'c')
 xlabel 'x', ylabel 'T'
```

Applet Activity. In the CA applet, observe the difference between the odd numbered coefficients of the $\mathcal{S}_9$, $\mathcal{P}_9$ and $\mathcal{I}_9$ approximations of the sign function (select sign from the Functions menu and set $N = 9$ using the slider at the bottom of the applet). On the Approximations menu, select the blue *interpolation, I* and then select the red *continuous, S*. On the Approximations menu select *plot coefficients*. There is no difference in the even numbered coefficients, as the sign function is odd. Thus the continuous even coefficients that are involved in the aliasing relation are all zero. The difference in the odd coefficients is due to aliasing. Make a similar comparison with the truncated discrete series by selecting the blue *discrete, P* from the approximations. Again there is a difference in the odd coefficients that is due to aliasing.

Now compare the two discrete approximations, $\mathcal{I}_9$ (blue *interpolation, I*) and $\mathcal{P}_9$ (red *discrete, P*). The coefficients are identical, but the approximations are different due to α_9 being halved in the interpolating approximation but not in the truncated series.

6   Rates of Convergence

Repeatedly integrating equation (5) by parts we get

$$\alpha_n = \frac{1}{n^m}\frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2}\ f^{(m)}(x)\ T_n(x)dx. \tag{19}$$
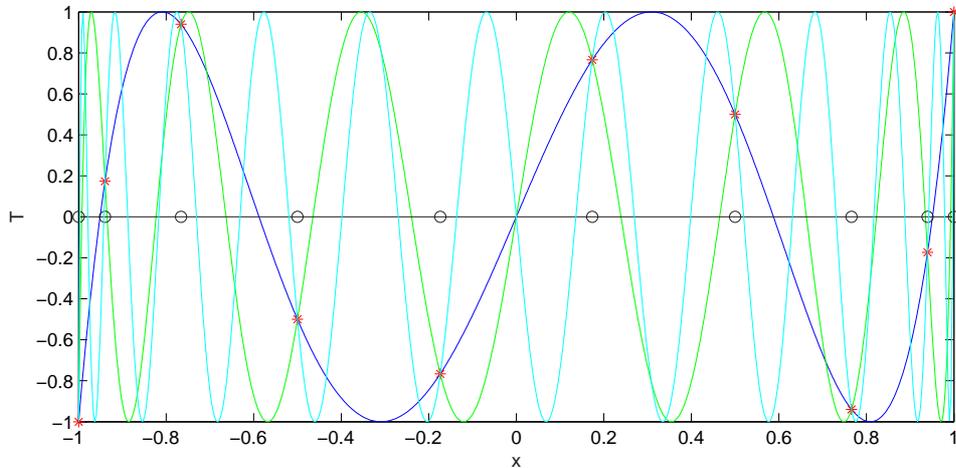
Figure 1: On the CGL grid (open black circles) for $N = 9$ T_5 is identical to T_{13} (green) and T_{23} (cyan). Points of intersection on the CGL grid are marked with red *'s.

Thus, if $f(x)$ is m-times ($m \geq 1$) continuously differentiable in $[-1, 1]$ the above integral will exist and we can conclude that

$$\alpha_n = \mathcal{O}(n^{-m}), \qquad n = 1, 2, \ldots. \tag{20}$$

If we make a careful choice of which definition of the integral to use, the same result can be shown to be true if $f(x)$ is $(m-1)$-times differentiable a.e. (almost everywhere) in $[-1, 1]$ with its $(m-1)^{th}$ derivative of bounded variation in $[-1, 1]$.

Since the absolute value of each T_k is bounded above by one on $[-1, 1]$, it follows that the truncation error for the continuous expansion is bounded by the sum of the absolute value of the neglected coefficients:

$$|f(x) - \mathcal{S}_N(x)| \leq \sum_{n=N+1}^{\infty} |\alpha_n|. \tag{21}$$

A similar bound, with an additional factor of two, holds for the interpolating partial sum:

$$|f(x) - \mathcal{I}_N(x)| \leq 2 \sum_{n=N+1}^{\infty} |\alpha_n| \tag{22}$$

for $x \in [-1, 1]$. From (20), (21), and (22) we conclude that

$$|f(x) - \mathcal{S}_N(x)| = \mathcal{O}(N^{-m}) \tag{23}$$

10

and

$$|f(x) - \mathcal{I}_N(x)| = \mathcal{O}(N^{-m}). \tag{24}$$

If f is infinitely differentiable the convergence is faster than $\mathcal{O}(N^{-m})$ no matter how large we take m. This is commonly termed *spectral accuracy* or *exponential accuracy*. If f can be extended to an analytic function in a suitable region of the complex plane, the pointwise error on $[-1, 1]$ can be shown to be

$$\mathcal{O}(r^{-N}) \tag{25}$$

for some $r > 1$ [14]. In figure 2 the rather slow decay rate of the error with increasing N is illustrated for the absolute value function (35) for which $m = 1$. This can be contrasted with the rapid spectral convergence of the infinitely smooth function (33) which is also illustrated in figure 2. Notice that the decay of error for the smooth function ceases at about $N = 140$. This is due to the accuracy of the representation of floating point numbers on the computer which limits accuracy to about 14 or 15 decimal places.

No matter what rate of decay the coefficients have, the convergence rate is only observed for $n > n_0$. Using an approximation with fewer than n_0 terms may result in a very bad approximation. For example, the decay rate of the coefficients of the infinitely smooth function in the applet is not yet evident for $N = 17$ and the approximation is very poor.
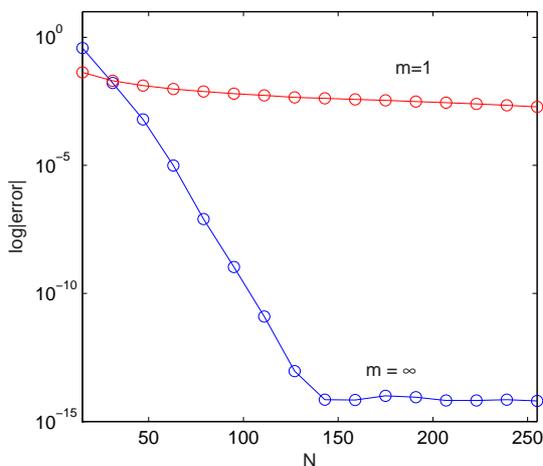

Figure 2: Convergence of an infinitely differentiable function $f_2(x) = e^{\cos(8x^3+1)}$ vs. convergence of a continuous function $f_5(x) = |x|$.

Applet Activity. Equation (19) allows us to conclude that if f is a polynomial of degree N, then $\alpha_n = 0$ for all $n > N$ since $f^{(n)}(x) = 0$ $n > N$. In the CP applet select the "7th degree polynomial" from the Functions menu. Use the slider at the bottom of the applet to set N to 9. From the Options menus check plot coefficients and semiLogY. Observe that $a_k = 0$ (to within machine precision) for $n > 7$.

If $m = 0$, i.e, f is discontinuous, the accuracy of the Chebyshev approximation methods reduces to $\mathcal{O}(1)$ near the discontinuity. Sufficiently far away from the discontinuity, the convergence will be slowed to $\mathcal{O}(N^{-1})$. Oscillations will be present near the discontinuity and they will not diminish as $N \to \infty$. Additionally, the oscillations will not even be localized near a discontinuity. This situation is referred to as the *Gibbs phenomenon*. A nice history of the Gibbs phenomenon can be found in [12].

Applet Activity. From the Functions menu select the *sign* function. From the Options menus uncheck plot coefficients and check semiLogY. Use the slider at the bottom of the applet to slowly change N from 10 to 256. Observe that the maximum amplitude of the overshoot at the discontinuity does not decrease with increasing N. Observe that sufficiently far away from the discontinuity that the oscillations are slowly decaying. Now check plot coefficients on the Options menu and again use the slider at the bottom of the applet to slowly change N from 10 to 256. Notice that the coefficients are decaying, but at a very slow rate. Spectral convergence has been lost due to the discontinuity. Select the "smooth" function from the Functions menu and compare how fast the coefficients of this function decay compared to the sign function.

7 Filters

Spectral filters can be used to enhance the decay rate of the Chebyshev coefficients [20] and to lessen the effects of the Gibbs phenomenon. The filtered Chebyshev approximation is

$$\mathcal{F}_N(x) = \sum_{n=0}^{N} \sigma\left(\frac{n}{N}\right) a_n T_n(x) \tag{26}$$

where σ is a spectral filter. A pth $(p > 1)$ order spectral filter is defined as a sufficiently smooth function satisfying

$$\sigma(0) = 1 \tag{27}$$
$$\sigma^{(m)}(0) = 0 \qquad m = 1, 2, \ldots, p-1 \tag{28}$$
$$\sigma^{(m)}(1) = 0 \qquad m = 0, 1, \ldots, p-1. \tag{29}$$

The convergence rate of the filtered approximation is determined solely by the order of the filter and the regularity of the function away from the point of discontinuity.

If p is chosen increasing with N, the filtered expansion recovers exponential accuracy away from a discontinuity. Assuming that $f(x)$ has a discontinuity at x_0 and setting $d(x) = x - x_0$, the estimate

$$|f(x) - \mathcal{F}_N(x)| \leq \frac{K}{d(x)^{p-1} N^{p-1}} \tag{30}$$

holds where K is a constant. If p is sufficiently large, and $d(x)$ not too small, the error goes to zero faster than any finite power of N, i.e. spectral accuracy is recovered. When x is close to a discontinuity the error increases. If $d(x) = \mathcal{O}(1/N)$ then the error estimate is $\mathcal{O}(1)$.

Many different filter functions are available, but perhaps the most versatile and widely used filter is the exponential filter

$$\sigma(\omega) = \exp(-\alpha\omega^{2p}) \quad p = 1, 2, \ldots \tag{31}$$

of order $2p$. In order for condition (29) to be satisfied, the parameter α is taken as $\alpha = -\ln\varepsilon_m$ where ε_m is defined as machine zero. On a 32-bit machine using double precision floating point operations, $\varepsilon_m = 2^{-52} \approx$ 2.2204e-16 and $ln(\varepsilon) \approx -36.0437$.

Applet Activity. The exponential filter is implemented in the CA applet. The default order of the filter is 4 ($p = 2$). Select the sign function from the Functions menu. From the Approximations menu select the blue interpolation and red filter options. From the options menu check semiLogY and uncheck connect. Use the slider to increase N and observe the rapid decrease in the error of the filtered approximation away from the discontinuity. The filter has restored spectral accuracy at points sufficiently far away from the discontinuity. Next, check plot coefficients on the options menu and compare the filtered and unfiltered coefficients. Now, display the parameters dialog from the options menu and enter 1 in the filter order box

to change the order of the filter to 2. Repeat the above experiments. Observe how the sharp front at the discontinuity is rounded or smeared in the filtered approximation by the low order filter. Enter 4 in the filter order box to change the order of the filter to 8 and repeat. What do you observe?

Applet Activity. Select the absolute value function from the Functions menu and repeat the previous applet activity.

Applet Activity. The EP applet illustrates the strength of the damping applied in equation (26) to the coefficients a_k from $k = 0, 1, \ldots, N$ for filters of order 2 to 32. The slider at the bottom of the applet can be used to change the order of the filter. Observe that as the order of the filter increases that less damping is applied to the coefficients with small k.

8 Applet Example Functions

The example functions included in the applet are the *sign* function ($m = 0$)

$$f_1(x) = \begin{cases} 1 & -1 \leq x \leq 0, \\ -1 & 0 < x \leq 1, \end{cases} \tag{32}$$

the "*smooth*" function ($m = \infty$)

$$f_2(x) = e^{\cos(8x^3+1)}, \tag{33}$$

the *square root* function ($m = 1$)

$$f_4(x) = \sqrt{1 - x^2}, \tag{34}$$

the *absolute value* function ($m = 1$)

$$f_5(x) = |x|, \tag{35}$$

and a *seventh degree polynomial*

$$f_6(x) = x^7 - 2x^6 + x + 3. \tag{36}$$

14

9   Current Research Area

Chebyshev approximation is an old and rich subject. However, many areas that employ Chebyshev polynomials have open questions that have attracted the attention of current researchers. One example is pseudospectral methods for the numerical solution of partial differential equations (PDEs). Chebyshev pseudospectral methods, which are based on the interpolating Chebyshev approximation (12), are well established as powerful methods for the numerical solution of PDEs with sufficiently smooth solutions.

Interpolation means that $f(x)$, the function that is approximated, is a known function. The terms *collocation* and *pseudospectral* are applied to global polynomial interpolatory methods for solving differential equations for an unknown function $f(x)$. Detailed information on pseudospectral methods may be found in the standard references [3, 6, 7, 9, 10, 19].

Many important PDEs have discontinuous (or nearly discontinuous) solutions. See the article [16] for a discussion of one such class of PDEs, nonlinear hyperbolic conservation laws. In these cases, the Chebyshev pseudospectral method produces approximations that are contaminated with Gibbs oscillations and suffer from the corresponding loss of spectral accuracy, just like the Chebyshev interpolation methods that the pseudospectral methods are based on.

An active research area is the development of postprocessing methods to remove the Gibbs oscillations from PDE solutions and to restore spectral accuracy. Spectral filters may be used but they perform poorly in the neighborhood of discontinuities. More sophisticated methods that do better in the area of discontinuities, but they may need to know the exact location of the discontinuities. The methods include Spectral Mollification, Gegenbauer Reconstruction [11], Padé Filtering, and Digital Total Variation Filtering. Several postprocessing methods with applications are discussed in [17] with supporting web material at http://www.scottsarra.org/signal/signal.html. The ultimate goal is a "black box" postprocessing algorithm, which can be given an oscillatory PDE solution and return a postprocessed solution with spectral accuracy restored.

10   Further Explorations

In addition to the exponential filter, other postprocessing methods for lessening the effects of the Gibbs phenomenon exist. Explore some of them which include:

1. Other spectral filters. See [2].

2. Reprojection methods. Reprojection methods work by projecting the slowly converging Chebyshev approximation onto a Gibbs complementary basis in which the convergence is faster. See [8, 11, 17].

3. Padé based reconstruction. Padé methods reconstruct the Chebyshev polynomial approximation as a rational approximation [13].

4. Digital Total Variation (DTV) filtering. DTV methods which were developed in image processing have been used to postprocess Chebyshev approximations. See [18].

11 Summary

Chebyshev approximation and its relation to polynomial interpolation at equidistant nodes has been discussed. We have illustrated how the Chebyshev methods approximate with spectral accuracy for sufficiently smooth functions and how less smoothness slows down convergence. We have illustrated how the presence of a discontinuity leads to lack of convergence at the discontinuity and leads to slowed convergence away from the discontinuity. We have described the Gibbs phenomenon which is characterized by a lack of or slow convergence as well as non-physical oscillations. Spectral filtering was discussed as a method used to lessen the effects of the Gibbs phenomenon and to restore spectral accuracy sufficiently far away from a discontinuity. Postprocessing methods to lessen the effects of the Gibbs oscillations are an active research area which would be an excellent topic for undergraduate research or as the topic of a Masters thesis.

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